8.A. Regular heptagons and cubic polynomials

In the main notes for this unit we mentioned that F. Vieta had given a method for constructing a regular 7-sided polygon (a *Heptagon*) based upon the fact that $\cos(360/7)^\circ$ is the root of a cubic polynomial with integral coefficients. We shall explain this further here.

To simplify the formulas we shall denote $(360/7)^\circ$ by Θ.

If $\xi = \cos\,\Theta + i\,\sin\,\Theta$ then using the polar forms of complex numbers one sees that

$$\xi^7 \quad = \quad \cos\,7\Theta + i\,\sin\,7\Theta \quad = \quad 1$$

and hence ξ is a root of the polynomial

$$X^7 - 1 \quad = \quad (X-1)\cdot(X^6 + X^5 + X^4 + X^3 + X^2 + X + 1)$$

and since $\xi \neq 1$ it follows that ξ must be a root of the second factor. Similarly, one sees that the conjugate $\overline{\xi} = \cos\,\Theta - i\,\sin\,\Theta$ must also be a root of the same polynomial. Adding these we obtain the equation

$$(\xi+\overline{\xi})^6 + (\xi+\overline{\xi})^5 + (\xi+\overline{\xi})^4 + (\xi+\overline{\xi})^3 + (\xi+\overline{\xi})^2 + (\xi+\overline{\xi}) + 2 \quad = \quad 0$$

and if we combine these with the identity

$$(\xi+\overline{\xi})^k \quad = \quad 2\,\cos\,k\Theta$$

we obtain the identity

$$2\,\cos\,6\Theta + 2\,\cos\,5\Theta + 2\,\cos\,4\Theta + 2\,\cos\,3\Theta + 2\,\cos\,2\Theta + 2\,\cos\,\Theta + 2 \quad = \quad 0\;.$$

Since $7\Theta = 360^\circ$ it follows that $\cos\,k\Theta = \cos\,(7-k)\Theta$, and therefore we have

$$\cos\,6\Theta \quad = \quad \cos\,\Theta \quad , \quad \cos\,5\Theta \quad = \quad \cos\,2\Theta \quad , \quad \cos\,3\Theta \quad = \quad \cos\,4\Theta\;.$$

If we use these equations to simplify the left hand side divide the main equation by 2 we obtain the identity

$$2\,\cos\,3\Theta + 2\,\cos\,2\Theta + 2\,\cos\,\Theta + 1 \quad = \quad 0\;.$$

We can now use the facts that (i) $\cos\,2x$ is a quadratic polynomial in $\cos x$ with integral coefficients, (ii) $\cos\,3x$ is a cubic polynomial in $\cos x$ with integral coefficients, to conclude that $\cos\,\Theta$ satisfies a cubic polynomial with integral coefficients. In fact, one can substitute using the identities

$$\cos\,2x \quad = \quad 2\,\cos^2\,x - 1 \qquad , \qquad \cos\,3x \quad = \quad 4\,\cos^3\,x - 3\,cos\,x$$

to write down this cubic polynomial explicitly:

$$8\,\cos^3\,\Theta \;+\; 4\,\cos^2\,\Theta \;+\; 4\,\cos\,\Theta \;-\; 1 \quad = \quad 0$$

If we write $u = 2\cos\Theta$ then this reduces to the moic cubic polynomial $u^3 + u^2 - 2u - 1 = 0$. One geometric way of finding a root of this equation is to make a linear change of variables to eliminate the quadratic term and then use Omar Khayyam's method for finding a root of the new polynomial using a circle and some other conic (either a hyperbola or a parabola).■