**NAME:** ______________________________

**Mathematics 153, Winter 2005, Examination 1**

*Point values are indicated in brackets.*

1. [*20 points*] ($i$) Let $a$ and $b$ be positive integers, and let $d$ be the greatest common divisor of $a$ and $b$. Prove that $d$ divides every integer of the form $s\,a\ +\ t\,b$ where $s$ and $t$ are integers.

## SOLUTION.

If $d$ divides both numbers then $a = u\,d$ and $b = v\,d$ for some integers $u$ and $v$. Therefore

$$s\,a\ +\ t\,b\ \ =\ \ s\,ud\ +\ t\,vd\ \ =\ \ (s\,u\ +\ t\,v)\cdot d$$

so that $d$ also divides $s\,a\ +\ t\,b$.∎

($ii$) Using the preceding part of the problem, show that two consecutive odd integers are relatively prime. [*Hints:* Why is $d$ an odd integer? What is the difference between two consecutive odd integers?]

## SOLUTION.

Suppose that $d$ is the greatest common divisor of the numbers, and write them as $2k + 1$ and $2k + 3$. If $d$ divides both, then $d$ divides their difference which is 2. But if $d$ divides either then $d$ must be odd. Since the only odd positive integer dividing 2 is 1, it follows that $d = 1$ and the original pair of odd integers is realtively prime.∎

2. [*25 points*] Suppose that $n$ is an integer.

($i$) If $n$ has the form $3q + r$ where $r = 1$ or 2, show that $n^2 = 3k + 1$ for some integer $k$.

**SOLUTION.**

If $n = 3q+1$ then $n^2 = 9q^2+6q+1 = 3(3q^2+2q)+1$, and if $n = 3q+2$ then $n^2 = 9q^2+12q+4 = 3(3q^2 + 6q + 1) + 1$.■

($ii$) Prove that the equation $x^2 = 3y + 2$ has no solution such that $x$ and $y$ are both integers. [*Hint:* Suppose $x = 3q + r$ where $r$ is 0, 1 or 2. Show that $x^2 = 3k + s$ where $s = 0$ or 1.]

**SOLUTION.**

The first part shows that there are no solutions of the form $3q + r$ where $r = 1$ or 2. The only other possibility would be solutions of the form $3q$. But $(3q)^2 = 9q^2$ is divisible by 3 and thus cannot have the form $3y + 2$ either. Since every integer has the form $3q + r$ where $r$ is 0, 1 or 3, it follows that the square of an integer $x$ never has the form $3y + 2$.■

3. [*20 points*] Suppose we are given a circle $C$ in the coordiante plane with center $(0, 2a)$ and radius $a$. Let $S$ be the surface of revolution obtained by rotating $C$ about the $x$-axis and let $T$ be the solid of revolution formed by rotating the regiou bounded by $C$ about the $x$-axis. Find the surface area of $S$ and the volume of $T$ using the Pappus Centroid Theorem.

## SOLUTION.

Note first that the centroid of the circle is $(0, 2a)$, so that the distance from the centroid to the $x$-axis is $2a$ and the distance traveled by the centroid when rotated about the $x$-axis is $4\pi\, a$. Let $D$ be the disk that $C$ bounds. Then by the Pappus Centroid Theorem(s) we have the following:

$$\text{area}(S) \quad = \quad \text{length}(C) \cdot 4\pi\, a \quad = \quad (2\pi\, a) \cdot (4\pi\, a) \quad = \quad 8\,\pi^2\, a^2$$

$$\text{volume}(T) \quad = \quad \text{area}(D) \cdot 4\pi\, a \quad = \quad (\pi\, a^2) \cdot (4\pi\, a) \quad = \quad 4\,\pi^2\, a^3$$

4. [*35 points*] For each of the topics listed below, match the name of a person who contributed significantly to that topic using the letter key indicated below. No name should be used more than once.

____ Computations of areas and volumes

____ Criterion for finding amicable pairs of numbers

____ Extensive tables of trigonometric functions

____ Geometric solutions of cubic equations

____ Prime number sieve

____ Properties of conic sections

____ Shorthand non-rhetorical notation for algebraic expressions

____ Use of negative numbers

**A :** Al-Khwarizimi
**B :** Apollonius
**C :** Archimedes
**D :** Aryhbhatta
**E :** Brahmagupta
**F :** Claudius Ptolemy
**G :** Diophantus
**H :** Eratosthenes
**I :** Menelaus
**J :** Omar Khayyam
**K :** Proclus
**L :** Thabit ibn Qurra

**SOLUTION.**

C
L
A or D or F or I
J
H
B
E or G
E