SOLUTIONS TO EXERCISES FOR

MATHEMATICS 153 — Assignment 3

Spring 2005

PROBLEMS FROM BURTON, p. 163

12. There is a similarity of triangles $\Delta ADC \sim \Delta ABD$ because $\angle CAD = \angle DAB$ and $|\angle ADC| = |\angle ABD| = 90°$. Likewise, there is a similarity of triangles $\Delta ADC \sim \Delta DBC$ because $\angle DCA = \angle BCD$ and $|\angle ADC| = |\angle DBC| = 90°$. Since corresponding sides under similarities have proportional length, this means that

$$\frac{|AB|}{|BD|} \quad = \quad \frac{|DB|}{|BC|}$$

and if we substitue the values for the lengths of segments we conclude that

$$\frac{n}{|BD|} \quad = \quad \frac{|DB|}{1}$$

which immediately implies $|BD| = \sqrt{n}$.■

PROBLEMS FROM BURTON, p. 175

1. (b) If $a|b$ then $b = x\,a$, and if $a|c$ then $c = y\,a$. Therefore $bc = xy\,a^2$, which implies $a^2 \mid bc$.■

(d) If $a|a+b$ then $a+b = x\,a$, and therefore

$$b \quad = \quad (a+b) \; - \; a \quad = \quad x\,a \; - \; a \quad = \quad (x-1)\,a$$

which implies $a|b$.■

10. (b) Expand $(2n+1)^2 = 4\,n^2 + 4\,n + 1$. The assertion in the exercise will be true if $4\,n^2 + 4\,n$ is divisible by 8, and this in turn is true if $n(n+1)$ is even. But the latter is true because one of the numbers n and $n+1$ is always even.■

16. (c) One way to begin is to find the GCD by factoring the numbers explicitly. We have $119 = 7 \cdot 17$ and $272 = 16 \cdot 17$, so the GCD is 17. To express 17 as an integral linear combination of 119 and 272 we use long division:

$$272 \quad = \quad 2 \times 119 \; + \; 34$$

$$119 \quad = \quad 3 \times 34 \; + \; 17$$

$$34 \quad = \quad 2 \times 17 \; + \; 0$$

We now work backwards from these equations:

$$17 \quad = \quad 2 \times 119 \; - \; 3 \times 34$$

$$3 \times 34 \;=\; 3 \times 272 \;-\; 6 \times 119$$

Combining these equations we obtain the desired equation:

$$17 \;=\; 119 \;-\; 3 \times 272 \;+\; 6 \times 119 \;=\; 7 \times 119 \;-\; 3 \times 272$$

One can check this by direct computation of the right hand side.■

(d) Here is what the Euclidean algorithm yields in this case:

$$2378 \;=\; 1 \times 1769 \;+\; 609$$

$$1769 \;=\; 2 \times 609 \;+\; 551$$

$$609 \;=\; 1 \times 551 \;+\; 58$$

$$551 \;=\; 9 \times 58 \;+\; 29$$

$$58 \;=\; 2 \times 29 \;+\; 0$$

This means that the greatest common divisor is 29. Once again we work backwards:

$$29 \;=\; 551 \;-\; 9 \times 58 \;=\; 551 \;-\; 9 \times (609 \;-\; 551) \;=\; 10 \times 551 \;-\; 9 \times 609 \;=$$

$$10 \times (1769 \;-\; 2 \times 609) \;-\; 9 \times 609 \;=\; 10 \times 1769 \;-\; 29 \times 609 \;=$$

$$10 \times 1769 \;-\; 29 \times (2378 \;-\; 1769) \;=\; 39 \times 1769 \;-\; 29 \times 2378$$

One can check this by direct computation of the right hand side.■

26. Write $p = 6m + r$ where $0 \leq r \leq 5$. We claim that $r = 1$ or 5, so we need to show that if $r = 0, 2, 3, 4$ then p is not prime unless $p = 2$ or 3. If $r = 0, 2, 4$ then p is a sum of two even numbers and hence even. On the other hand, if $r = 3$ then $p = 3(2m + 1)$ and the only way that p can be prime is if $2m + 1 = 1$, which means that $p = 3$.■

ADDITIONAL EXERCISES FROM `additional3.*`

1. The measure $|\angle A|$ can be either $|\angle D|$ or $|\angle B|$ but **NOT** $|\angle C|$. One can realize the possibility $|\angle A| = |\angle D|$ by starting with a rectangle $ABXD$ and taking C to be the midpoint of the segment $[XD]$. The possibility $|\angle A| = |\angle B|$ is realizable by taking an equilateral triangle XAB, and letting C and D be the midpoints of $[XB]$ and $[XC]$ respectively. Since the line joining the midpoints of two sides of a triangle is parallel to the third side, the line CD is parallel to AB.

In any trapezoid as described in the problem we know that the angles $\angle A$ and $\angle D$ are supplementary. If $|\angle A| = |\angle C|$ this means that angles $\angle C$ and $\angle D$ are also supplementary. The latter implies that AD and BC are parallel, contradicting our assumption that they were not. Therefore we must have $|\angle A| \neq |\angle C|$.■

2. The correspondence $\Delta ADC \leftrightarrow \Delta ADB$ is a Side-Side-Angle correspondence because $[AD] = [AD]$, $\angle ADC = \angle ADB$ (the rays $[DB$ and $[DC$ are equal), and $|AC| = |AB|$ (because ΔABC is equilateral).

We shall derive the values for the angle measures using a proof in a modified traditional format.

(1) $|BC| = |BD|$ (*Since C is the midpoint of* $[BD]$.)
(2) $|AC| = |BC|$ (*Since ΔABC is equilateral.*)

(3) $|AC| = |BD|$ (*Since* $u = v$ *and* $v = w \Longrightarrow u = w$.)
(4) $|\angle CAD| = |\angle CDA|$ (*Since* ΔCAD *is isosceles.*)
(5) $|\angle ACD| = 120°$ (*It is supplementary to the* $60°$ *angle* $\angle ACB$.)
(6) $|\angle CAD| = |\angle CDA| = 30°$. (*The sum of the angle measures for* ΔCAD *is* $180°$ *plus the previous two steps.*)
(7) $|\angle ADC| = |\angle ADB| = 30°$. ($\angle CDA = \angle ADC = \angle ADB$.)
(8) $|\angle BAD| = |\angle BAC| + |\angle CAB|$ (*Additivity of angle measures.*)
(9) $|\angle BAC| = 60°$. (*Since* ΔABC *is equilateral and hence equiangular.*)
(10) $|\angle BAD| = 90°$ (*Combine the last three steps.*)

3. (a) Condition [3] implies $x = 60°$ and one can combine this with [2] to conclude that $a = b = c$, contradicting $a > b$ in the given data.■

(b) Condition [4] and $9^2 < 6^2 + 7^2 \Longrightarrow z < 90°$, contradicting the given condition $s = 93°$.■

(c) Condition [1] implies $a + b > c$ but the given data satisfy $a + b = c$.■

4. (a) For an equilateral all three points are the same, and they lie inside the triangle.■

(b) For an isosceles right trangle the centroid lies inside the triangle, the orthocenter is at the right angle vertex, and the circumcenter is the midpoint of the hypotenuse.■

(c) In this case the centroid still lies inside the triangle while both of the other points lie outside the triangle.

Since this case is less transparent than the other two, we shall discuss it more explicitly using coordinates. An example of such a triangle is ΔABC where $A = (0, 1)$, $B = (\sqrt{3}, 0)$ and $C = (-\sqrt{3}, 0)$. The centroid has coordinates $(0, \frac{1}{3})$, the circumcenter has coordinates $(0, -1)$, and the orthocenter has coordinates $(0, \sqrt{3})$.■