# SOLUTIONS TO EXERCISES FOR

# MATHEMATICS 153 — Assignment 4

**Spring 2005**

### PROBLEM FROM BURTON, p. 185

**7.** The file `quadcirc.*`, where $* =$ `ps` or `pdf`, contains a picture of a quadrilateral that is inscribed in a circle and circumscribed about another circle. In this picture the vertices are $A, B, C, D$ and the points of tangency with the smaller circle are

$E$, which is between $A$ and $B$,
$F$, which is between $B$ and $C$,
$G$, which is between $C$ and $D$, and
$H$, which is between $D$ and $A$.

The lengths of the sides will be denoted by $a = |AB|$, $b = |BC|$, $c = |CD|$ and $d = |DA|$. Since the lengths of the two tangent segments from an external point are equal (see the online link cited in the assignment) we have $|AH| = |AE| = w$, $|BE| = |BF| = x$, $|CF| = |CG| = y$, and $|DG| = |DH| = z$. We then have $a = x + y$, $b = y + z$, $c = z + w$, and $d = w + x$.

If $s = \frac{1}{2}(a + b + c + d)$, then the formulas of the previous paragraph imply $s = x + y + z + w$. Therefore we have $s - a = z + w = c$, $s - b = w + x = d$, $s - c = x + y = a$, and $s - d = y + z = b$.

As noted in a previous exercise, the area bounded by the inscribed quadrilateral is

$$\sqrt{(s-a)(s-b)(s-c)(s-d)}$$

and by the identities of the previous paragraph this is equal to $\sqrt{a\,b\,c\,d}$.■

### PROBLEMS FROM BURTON, p. 199

**1.** (*b*) The formula says that the area of the pointed pieces of the cone's surface is equal to $\pi K^2$, where $K^2 = r\,s$ with $r$ equal to the radius of the cone's base and $s$ is the slant height, which is the length of the segment joining the nappe of the cone to a point on the base.

For the sake of completeness we shall indicate how to compute the surface area for a surface of revolution by integral calculus. The standard textbook formula for surface areas states that the area of surface of revolution generated by a graph curve $y = f(x)$ is given by the following expression:

$$2\pi \cdot \int_a^b f(x) \cdot \sqrt{1 + [f'(x)]^2}\, dx$$

If we apply this to a cone, the function in question has the form $y = mx$ for some $m > 0$ and the limits of integration are from $x = 0$ to $x = h$, where $h$ is the height of the cone. If we compute the area of the portion of the cone using the formula, we see that it is equal to

$$2\pi \cdot \int_0^h m\sqrt{1+m^2}\, x\, dx \quad = \quad \pi\, m\sqrt{1+m^2}\, h^2$$

where $r = m\,h$. By the Pythagorean Theorem we know that the slant height $s$ is equal to $\sqrt{r^2 + h^2}$, and therefore it follows that

$$1 \;+\; m^2 \quad=\quad \frac{r^2 + h^2}{h^2}$$

and therefore we also have that the area of the piece of the cone is equal to

$$\pi\, m\, \sqrt{1+m^2}\, h^2 \quad=\quad \pi\, h\, \sqrt{1+m^2}\, m\, h \quad=\quad \pi\, h\, \sqrt{\frac{r^2+h^2}{h^2}}\, m\, h \quad=\quad \pi\, s\, r$$

which is exactly the statement in Archimedes' result.■

(*d*) A great circle is one whose center is the same as the center of the sphere. Therefore if $r$ is the common radius we know that the area bounded by the great circle is $\pi\, r^2$ and the surface area of the sphere is $4\,\pi\, r^2$. Therefore the surface area of the sphere is four times the surface area bounded by a great circle on that sphere.■

**4.** The Pythagorean Theorem implies (*a*):

$$|OT|^2 \quad=\quad |OR|^2 \;-\; |TR|^2 \quad=\quad 1 \;-\; \frac{S_n^2}{4} \quad=\quad \frac{4 - S_n^2}{4}$$

The additivity formula $1 = |QT| + |OT|$ implies (*b*):

$$|QT|^2 \quad=\quad (1 \;-\; |OT|)^2 \quad=\quad \left(1 \;-\; \frac{\sqrt{1 - S_n^2}}{2}\right)^2$$

The second equation follows by substituting the formula for $|OT|^2$ in the first equation and simplifying the resulting expression.

Finally, the third equation follows by applying the Pythagorean Theorem to right triangle $RTQ$, substituting for $|RT|^2$ using (*b*), and also using the given fact that $|QR| = S_{2n}$:

$$S_{2n}^2 \quad=\quad \left(1 \;-\; \frac{\sqrt{1 - S_n^2}}{2}\right)^2 \;+\; \frac{S_n^2}{4} \quad=$$

$$1 \;+\; \frac{4 - S_n^2}{4} \;-\; \sqrt{4 - S_n^2} \;+\; \frac{S_n^2}{4} \quad=\quad 2 \;-\; \sqrt{4 - S_n^2} \;\;\blacksquare$$