SOLUTIONS TO EXERCISES FOR

MATHEMATICS 153 — Assignment 4

Spring 2005

PROBLEM FROM BURTON, p. 220

13. By the theorem on pages 215–216 of Burton, the equation $ax + by = c$ with integral coefficients has integral solutions if and only if the greatest common divisor d for a also divides c.

(a) The greatest common divisor of $6 = 2 \times 3$ and $51 = 17 \times 3$ is 3, but 22 is not divisible by 3, so there is no integral solution for the equation.■

(b) The greatest common divisor of $14 = 7 \times 2$ and $33 = 3 \times 11$ is 1, so the equation does have integral solutions.■

(c) The greatest common divisor of $14 = 7 \times 2$ and $35 = 7 \times 5$ is 7, and $91 = 13 \times 7$, so the equation does have integral solutions.■

PROBLEMS FROM BURTON, p. 226

1. (b) Use the hint and part (a). We have a polynomial $x^3 + bx^2 + cx + d$ with a root r/s, where r and s are relatively prime integers, and therefore s divides $a = 1$ and r divides d. This means that $\pm r$ is a root of the equation, and by the previous sentence we know that $r|d$.■

2. (b) By the results of the preceding exercise a rational root must have the form r/s where $r = \pm 1$ and s is equal to 2^m for some m satisfying $0 \leq m \leq 5$. In fact, $\frac{1}{2}$ and $-\frac{1}{4}$ are the roots of this polynomial, and the latter has multiplicity 2 (in other words, $(4x+1)^2$ divides the polynomial).■

(d) Every rational root of this monic polynomial must be an integer and a divisors of the constant term 24, and hence the only possibilities are ± 1, ± 2, ± 3, ± 4, ± 6, ± 8, ± 12, and ± 24. Now the polynomial $p(x) = x^3 - 7x^2 + 20x - 24$ is negative for $x < 0$, so we can narrow down the possibilities to the positive divisors of 24. Now $p(x) > 0$ if $x \geq 8$ and this reduces the options to 1, 2, 3, 4 and 6. One can then check directly that 3 is the only root.■