# SOLUTIONS TO EXERCISES FOR

# MATHEMATICS 153 — Assignment 6

**Spring 2005**

## PROBLEM FROM BURTON, p. 243

**7.** ($a$) In a problem like this it is always useful to draw rough sketches of the curves in question. The hyperbola $y^2 + cx = x^2$ may be rewritten in the form

$$(x - \tfrac{1}{2}c)^2 \ - \ y^2 \ = \ \tfrac{1}{4}c^2$$

which shows that its asymptotes are the lines $y = \pm\,(x - \frac{1}{2}c$ which meets the $x$-axis at $(0,0)$ and $(c,0)$. Note that the origin is the only point on either curve whose $x$-coordinate is equal to zero.

If we are given a point that lies on both curves with $x \neq 0$ then we have

$$x^2 \ = \ y^2 \ + \ c\,x \ = \ b^{-2}x^4 \ + \ c\,x$$

and if we cancel $x$ from both sides and multiply by $b^2$ we obtain

$$b^2x \ = \ x^3 \ + \ b\,c\,x$$

so that $x$ solves the original cubic equation.■

## ADDITIONAL PROBLEMS FROM `math153update7.pdf`

**1.** The surface of revolution obtained from the wire is the sphere of radius 1. Therefore the Pappus Centroid Theorem says that the area of the surface of the sphere is the length of the wire times the circumference of a circle of radius $\overline{y}$ where $\overline{y}$ is the $y$-coordinate of the centroid of the wire.

We know that the surface area is $4\,\pi$ and the length of the wire is $2\pi$, and therefore by the Pappus Centroid Theorem we have

$$4\,\pi \ = \ )(2\,\pi)\cdot(2\,\pi\,\overline{y}$$

and if we solve this for $\overline{y}$ we obtain $\overline{y} = 1/\pi$.

Similarly, the solid of revolution obtained from the half disk is a sphere of radius 1 and therefore its volume is $\frac{4}{3}\,\pi$. In this case the Pappus Centroid Theorem says that the volue of the solid sphere is the area of the semicircular disk times the circumference of a circle of radius $\overline{y}$ where $\overline{y}$ is the $y$-coordinate of the centroid of the half disk.

Since the area of the half disk is $\frac{1}{2}\,\pi$ it follows that

$$\tfrac{4}{3}\pi \ = \ )(\tfrac{1}{2}\,\pi)\cdot(2\,\pi\,\overline{y}$$

and if we solve this for $\overline{y}$ we obtain $\overline{y} = 4/3\,\pi$.

Therefore the inequality

$$\frac{4}{3\pi} \quad > \quad \frac{1}{\pi}$$

implies that the centroid of the semicircular wire is closer to the center of the circle than the centroid of the half disk.■

**2.** If we rotate the half ellipse about the $x$-axis we obtain an ellipse whose principal axes have lengths $a$, $b$ and $a$ in the $x$, $y$ and $z$ directions.

Before proceeding we note that the area formula for the ellpse also works if the lengths of the major and minor axes are $b$ and $a$ respectively; in fact it would probably be better to say simply that $a$ and $b$ are supposed to be the lengths of the principal axes.

The area of half the ellipse is $\frac{1}{2}\pi a b$, so by the Pappus Centroid Theorem we have

$$\tfrac{4}{3}\pi a^2 b \quad = \quad \left(\tfrac{1}{2}\pi a b\right) \cdot (2\pi \overline{y}$$

and if one solves this equation one finds that

$$\overline{y} \quad = \quad \frac{4a}{3\pi}$$

is the $y$-coordinate for the centroid.■

## SELECTED REVIEW PROBLEMS FROM BURTON, p. 220

**6.** Following the hint we want a number $r = 9 - x^2$ such that $21 - r = x^2 + 12$ is also a perfect square over the rationals. If we write $x = a/c$ and the perfect square as $b^2/c^2$, then clearing fractions yields the equation

$$a^2 + 12c^2 \quad = \quad b^2$$

where $a, b, c$ are supposed to be positive integers. When can this happen? It cannot happen if $c = 1$, but if $c = 2$ and $a = 1$ then we have

$$1 + 12c^2 \quad = \quad 1 + 48 \quad = \quad 49 \quad = \quad 7^2$$

which implies that $x^2 = \frac{1}{4}$ so that $r = \frac{35}{4}$ and we have

$$9 - r \quad = \quad \frac{1}{4}\,, \quad 21 - r \quad = \quad \frac{49}{4}$$

so that $9 - r$ and $21 - r$ are both perfect squares.■

**12.** Following the hint, consieder the Pythagorean triple $x^2-4,\ 4x,\ x^2+4$. Then the difference between the first and third is 8, which we know is a perfect cube. We also want $x^2+4-4x = (x-2)^2$ to be a perfect square. The latter is obviously a perfect square, so we really need to find a perfect square that is also a perfect cube. Now $64 = 8^2 = 4^3$ satisfies this requirement, and for this choice of $x$ we have $x = 10$, which meand that the triple in question is $96,\ 40,\ 104$. Note that there are many solutions, and in particular one could also take $x = 731$ because $729 = 27^2 = 9^3$.■