SOLUTIONS TO EXERCISES FOR

MATHEMATICS 153 — Assignment 7

Spring 2005

PROBLEMS FROM BURTON, p. 264

7. The condition implies $2n-1 = u^2$, and we have

$$n^2 \;=\; \big((n-1)+1\big)^2 \;=\; (n-1)^2 + 2(n-1) + 1 \;=\; (n-1)^2 + (2n-1) \;=\; (n-1)^2 + u^2\ .$$

Examples can be given using odd numbers that are perfect squares.■

11. Let x and y be the lengths of the legs of the right triangle, so that $b^2 = \frac{1}{2}xy$ and $a^2 = x^2 + y^2$. Then we have $a^2 + 4b^2 = (x+y)^2$ and $a^2 - 4b^2 = (x-y)^2$. If we solve these for x and y we obtain the values

$$\tfrac{1}{2}\left(\sqrt{a^2+4b^2} \;\pm\; \sqrt{a^2-4b^2}\right)$$

as asserted in the problem.■

PROBLEM FROM BURTON, p. 272

2. (a) Follow the hint to add the inequalities $F_{2i+1} = F_{2i} - F_{2i-2}$ for $1 \le i \le n$. The left hand side will be the sum of the odd Fibonacci numbers from F_1 to F_{2n-1}, and the right hand side is a telescoping sum whose value will be $F_{2n} - F_0 = F_{2n}$.■

(c) The formula is true for $n = 1$ so proceed by induction and assume it is true for $k \le n-1$. Then we have

$$\sum_{i=1}^{n} (-1)^{i+1} F_i \;=\; \sum_{i=1}^{n-1} (-1)^{i+1} F_i \;+\; (-1)^{n+1} F_n \;=\; (-1)^n F_{n-2} + 1 + (-1)^{n+1} F_n$$

and since $F_{n-1} = F_n - F_{n-2}$ the right hand side reduces to

$$1 + (-1)^{n+1} F_{n-1}$$

completing the inductive step of the proof.■

PROBLEMS FROM BURTON, p. 302

1. (b) Set $y = x+2$, so that

$$2 \;=\; x^3 + 6x^2 + 3x \;=\; (x+2)^3 - 12x - 8 + 3x \;=\; (x+2)^3 - 9(x+2) + 18 - 8 = y^3 - 9y + 10$$

and hence the original polynomial reduces to $y^3 - 9y + 8 = 0$. One root of this polynomial is $y = 1$, and $y^3 - 9y + 8 = (y-1)(y^2 + y - 8)$, so the remaining roots are $-\frac{1}{2}(1 \pm \sqrt{33})$. Using the inverse substitution $x = y - 2$ we see that the roots of the original polynomial are -1 and $-\frac{1}{2}(5 \pm \sqrt{33})$.■

3. (d) We shall try to use Cardan's formula as printed on page 302. In our situation $p = 9$ and $q = 12$. Thus

$$\frac{q}{2} \;=\; 6 \quad , \quad \frac{q^2}{4} \;=\; 36 \quad , \quad \frac{p^3}{27} \;=\; 27$$

, so the formula yields the root value

$$\sqrt[3]{6 + \sqrt{36-27}} \;+\; \sqrt[3]{6 - \sqrt{36-27}} \;=\; \sqrt[3]{6 + \sqrt{9}} \;+\; \sqrt[3]{6 - \sqrt{9}} \;=$$

$$\sqrt[3]{6+3} \;+\; \sqrt[3]{6-3} \;=\; \sqrt[3]{9} \;+\; \sqrt[3]{3}\ .$$