

NAME: _____

Mathematics 153, Spring 2005, Examination 2

Point values are indicated in brackets.

1. [20 points] (i) Let a and b be positive integers, and let d be the greatest common divisor of a and b . Prove that d divides every integer of the form $sa + tb$ where s and t are integers.

SOLUTION.

If d divides both numbers then $a = ud$ and $b = vd$ for some integers u and v . Therefore

$$sa + tb = sud + tvd = (su + tv) \cdot d$$

so that d also divides $sa + tb$.■

(ii) Using the preceding part of the problem, show that two consecutive odd integers are relatively prime. [Hints: Why is d an odd integer? What is the difference between two consecutive odd integers?]

SOLUTION.

Suppose that d is the greatest common divisor of the numbers, and write them as $2k + 1$ and $2k + 3$. If d divides both, then d divides their difference which is 2. But if d divides either then d must be odd. Since the only odd positive integer dividing 2 is 1, it follows that $d = 1$ and the original pair of odd integers is relatively prime.■

2. [25 points] Suppose that n is an integer.

(i) If n has the form $3q + r$ where $r = 1$ or 2 , show that $n^2 = 3k + 1$ for some integer k .

SOLUTION.

If $n = 3q + 1$ then $n^2 = 9q^2 + 6q + 1 = 3(3q^2 + 2q) + 1$, and if $n = 3q + 2$ then $n^2 = 9q^2 + 12q + 4 = 3(3q^2 + 6q + 1) + 1$. ■

(ii) Prove that the equation $x^2 = 3y + 2$ has no solution such that x and y are both integers.
[Hint: Suppose $x = 3q + r$ where r is 0, 1 or 2. Show that $x^2 = 3k + s$ where $s = 0$ or 1.]

SOLUTION.

The first part shows that there are no solutions of the form $3q + r$ where $r = 1$ or 2 . The only other possibility would be solutions of the form $3q$. But $(3q)^2 = 9q^2$ is divisible by 3 and thus cannot have the form $3y + 2$ either. Since every integer has the form $3q + r$ where r is 0, 1 or 2, it follows that the square of an integer x never has the form $3y + 2$. ■

3. [20 points] Suppose we are given a circle C in the coordinate plane with center $(0, 2a)$ and radius a . Let S be the surface of revolution obtained by rotating C about the x -axis and let T be the solid of revolution formed by rotating the region bounded by C about the x -axis. Find the surface area of S and the volume of T using the Pappus Centroid Theorem.

SOLUTION.

Note first that the centroid of the circle is $(0, 2a)$, so that the distance from the centroid to the x -axis is $2a$ and the distance traveled by the centroid when rotated about the x -axis is $4\pi a$. Let D be the disk that C bounds. Then by the Pappus Centroid Theorem(s) we have the following:

$$\text{area}(S) = \text{length}(C) \cdot 4\pi a = (2\pi a) \cdot (4\pi a) = 8\pi^2 a^2$$

$$\text{volume}(T) = \text{area}(D) \cdot 4\pi a = (\pi a^2) \cdot (4\pi a) = 4\pi^2 a^3$$

4. [35 points] For each of the topics listed below, match the name of a person who contributed significantly to that topic using the letter key indicated below. No name should be used more than once.

- ___ Computations of areas and volumes
- ___ Criterion for finding amicable pairs of numbers
- ___ Extensive tables of trigonometric functions
- ___ Geometric solutions of cubic equations
- ___ Prime number sieve
- ___ Properties of conic sections
- ___ Shorthand non-rhetorical notation for algebraic expressions
- ___ Use of negative numbers

- A** : Al-Khwarizmi
- B** : Apollonius
- C** : Archimedes
- D** : Aryabhata
- E** : Brahmagupta
- F** : Claudius Ptolemy
- G** : Diophantus
- H** : Eratosthenes
- I** : Menelaus
- J** : Omar Khayyam
- K** : Proclus
- L** : Thabit ibn Qurra

SOLUTION.

- C
- L
- A or D or F or I
- J
- H
- B
- E or G
- E