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Mathematics 153–001, Spring 2010, Final Examination

Answer Key

1. [*15 points*] (a) If a and b are integers and d is a third integer which divides both of them, show that d divides every integer of the form $sa + tb$, where s and t are some integers.

(b) For each positive integer k show that $4k + 1$ and $4k + 5$ have no common factors except ± 1. [*Hint:* Why does the first part imply that a nontrivial common factor divides 4 and hence must be even? Also, if d is even and d divides an integer m, can m be odd?]

(c) Find integers s and t such that $9s + 13t = 1$.

SOLUTION

(a) We may write $a = da'$ and $b = db'$ for some integers a' and b'. Then we have

$$sa + tb = sa'd + tb'd = (sa' + tb') \cdot d \ .$$

(b) Suppose that d is a nontrivial common factor of $4k + 1$ and $4k + 5$. Then by (a) we know that d also divides

$$(4k + 5) - (4k + 1) = 4$$

so that d must be ± 2 or ± 4. In particular, d must be even. However both $4k + 1$ and $4k + 5$ are odd, so no even number can be a common factor. Therefore the only common factors are ± 1.

(c) By the Euclidean Algorithm we have $13 = 9 \cdot 1 + 4$ and $9 = 4 \cdot 2 + 1$, and therefore we have

$$1 = 9 - 4 \cdot 2 = 9 - 2 \cdot (13 - 9) = 9 - 2 \cdot 13 + 2 \cdot 9 = 3 \cdot 9 - 2 \cdot 13 \ .$$

The general solution is given by $s = 3 + 13k$ and $t = -(2 + 9k)$ where k is some integer.

2. [*20 points*] Recall that the Fibonacci sequence is defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. — Let $\{ x_n \}$ be the sequence of continued fractions given by $x_2 = 1$ and

$$x_{n+1} \quad = \quad \frac{1}{1 + x_n}$$

for $n \geq 2$. Prove by induction that $x_n = F_{n-1}/F_n$.

SOLUTION

The proof goes by induction on n; the case $n = 2$ follows because $x_2 = 1 = F_1/F_2$. Suppose that the result is known for x_n, in which case we need to verify the result for $x_n + 1$. But

$$x_{n+1} \quad = \quad \frac{1}{1 + x_n} \quad = \quad \frac{1}{1 + (F_{n-1}/F_n)} \quad = \quad \frac{1}{(F_n + F_{n-1})/F_n}$$

and since $Fn + 1 = F_n + F_{n-1}$ the last expression is equal to

$$\frac{1}{F_{n+1}/F_n} \quad = \quad \frac{F_n}{F_{n+1}}$$

which is the formula given in the problem.

3. [*20 points*] (a) Using the identity

$$\cos 5\,\theta \quad = \quad 16\,\cos^5\theta \;-\; 20\,\cos^3\theta \;+\; 5\,\cos\theta$$

explain why the positive number $\cos 18°$ satisfies a fourth degree polynomial equation of the form $ax^4 + bx^2 + c = 0$, where a, b, c are integers and $a \neq 0$. [*Hint:* $18 \times 5 = ?$]

(b) If we take the quadratic polynomial $ay^2 + by + c$, where a, b, c are as in the first part of the problem, then it turns out that $\cos^2 18°$ is the larger of the quadratic polynomial's roots. Using this and the formula $\cos 2\alpha = 2\cos^2\alpha - 1$, find rational numbers p and q such that $\cos 36° = p + q\sqrt{5}$.

SOLUTION

(a) Since $90 = 18 \times 5$, it follows that $90° = 5\alpha$ and hence

$$0 \quad = \quad \cos 5\,\alpha \quad = \quad 16\,\cos^5\alpha \;-\; 20\,\cos^3\alpha \;+\; 5\,\cos\alpha\ .$$

Since $\cos\alpha > 0$ we can divide by $\cos\alpha$ and obtain the equation

$$0 \quad = \quad 16\,\cos^4\alpha \;-\; 20\,\cos^2\alpha \;+\; 5\ .$$

This is a quadratic equation in $\cos^2\alpha$.

(b) By (a) we know that $\cos^2 18°$ is a root of $16\,y^2 \;-\; 20\,y \;+\; 5 \;=\; 0$, and if we apply the Quadratic Formula to the latter we see that it has the following roots:

$$y \quad = \quad \frac{5 \;\pm\; \sqrt{5}}{8}$$

Since the larger root is the one with the plus sign, must have

$$\cos^2 18° \quad = \quad \frac{5 \;+\; \sqrt{5}}{8}\ .$$

If we now apply the double angle identity $\cos 2\alpha = 2\cos^2\alpha - 1$, we see that

$$\cos 36° \quad = \quad 2\cos^2 18° \;-\; 1 \quad = \quad \frac{1}{4} \;+\; \frac{1}{4}\cdot\sqrt{5}$$

so that both rational coefficients are $\frac{1}{4}$.

Note. Here is a brief explanation why $\cos^2 18°$ is the larger root of $ay^2 + by + c = 0$: We know that $\cos\theta$ is strictly decreasing from $0°$ to $90°$, and since $\cos^2 45° = \frac{1}{2}$ it follows that $\cos^2 18° > \frac{1}{2}$. Now the smaller root of the given quadratic polynomial will be less than $\frac{1}{2}$ because $\sqrt{5} > 1$, and therefore $\cos^2 18°$ must be the larger of the roots.

4. [*15 points*] (*a*) If $|x| < 1$ then the geometric series implies that $1/(1-x) = \sum_{n\geq 0} x^n$. Using term by term differentiation and some algebra, derive an infinite series expansion for $x/(1-x)^2$.

(*b*) Using the preceding, evaluate

$$\sum_{n=1}^{\infty} \frac{n}{3^n} \ .$$

SOLUTION

(*a*) Start with the geometric series

$$\sum_{n=0}^{\infty} x^n \quad = \quad \frac{1}{1-x}$$

which is valid for $|x| < 1$, and differentiate term by term to obtain the derivative of the original infinite series:

$$\sum_{n=1}^{\infty} nx^{n-1} \quad = \quad \frac{1}{(1-x)^2}$$

Multiply both sides by x to obtain the following equation, which is valid for $|x| < 1$:

$$\sum_{n=1}^{\infty} nx^n \quad = \quad \frac{x}{(1-x)^2}$$

(*b*) The series we want is given by letting $x = \frac{1}{3}$. Therefore we obtain the following:

$$\sum_{n=1}^{\infty} \frac{n}{3^n} \quad = \quad \left.\frac{x}{(1-x)^2}\right|_{x=1/3} \quad = \quad \frac{1/3}{(2/3)^2} \quad = \quad \frac{3}{4}$$

5. [*20 points*] Find a change of variables $y = x + a$ that will transform the cubic equation $x^3 + 3\,x^2 - x - 3 = 0$ to an equation of the form $y^3 + p\,y + q = 0$, and use this equation to find all the roots of p. [*Hint:* The new polynomial factors into linear polynomials very easily!]

SOLUTION

Set $y = x + 1$ or equivalently $x = y - 1$, so that we may rewrite the equation in terms of y as follows:

$$0 \;=\; (y-1)^3 \;+\; 3(y-1)^2 \;-\; (y-1) \;-\; 3 \;=$$
$$(y^3 - 3y^2 + 3y - 1) \;+\; (3y^2 - 6y + 3) \;+\; (-y + 1) \;-\; 3 \;=\; y^3 \;-\; 4y$$

This equation can easily be factored into $0 = y(y-2)(y+2)$.

The roots of this equation are $y = 0, \pm 2$, and the roots of the original equation are $x = y - 1 = 1, -1, -3$.

6. [*10 points*] Explain what it means to say that the sequence of perfect squares 1^2, 5^2, 7^2 form an arithmetic progression, and find a second such sequence beginning with 2^2.

SOLUTION

In an arithmetic progression the differences between consecutive terms are always the same. Since $5^2 - 1^2 = 24 = 7^2 - 5^2$, we have an arithmetic progression.

If we multiply all terms in an arithmetic progression by a fixed constant, we obtain another arithmetic progression, so the corresponding arithmetic progression is $4 \cdot 1^2 = 2^2$, $4 \cdot 5^2 = 10^2$ and $4 \cdot 7^2 = 14^2$.

7. [*15 points*] For each of the following, determine which (if either) came first or if they were nearly contemporary. Five correct answers will yield full credit, and extra credit is possible.

(a) Dedekind or Euler

(b) Napier or Lagrange

(c) Cardano or Tartaglia

(d) Daniel Bernoulli or Fourier

(e) Cauchy or Weierstrass

(f) Viète or Wallis

(g) Stevin or Torricelli

(h) Barrow or Gregory

SOLUTION

(a) Euler

(b) Napier

(c) [contemporaries]

(d) Daniel Bernoulli

(e) Cauchy

(f) Viéte

(g) Stevin

(h) [contemporaries]

8. [*10 points*] Put the following figures from Greek mathematics in historical order.

[A] Archimedes

[B] Diophantus

[C] Eudoxus

[D] Euclid

[E] Hippocrates of Chios

[F] Thales

SOLUTION

F E C D A B

9. [*30 points*] Match any five of the topics with two mathematicians who made contributions to it. There may be several correct matches in many cases, but only two should be given (with zero credit if more than three are given). Extra credit will be granted if more than five topics are correctly matched.

____ Coordinate geometry

____ Modern symbolic mathematical notation

____ Results leading to the development of calculus

____ Logarithms, trigonometric functions, or decimal systems

____ Roots of third degree polynomial equations

A : Barrow
B : Bombelli
C : Briggs
D : Cardano
E : Descartes
F : Fermat
G : Gregory
H : Napier
I : Recorde
J : Roberval
K : Schooten
L : Torricelli
M : Viéte
N : Wallis
O : Widman

SOLUTION

Coordinate geometry **A D E F K N**

Modern symbolic mathematical notation **B E F I K M O**

Results leading to the development of calculus **A E F G J L N**

Logarithms, trigonometric functions, or decimal systems **C G H M N**

Roots of third degree polynomial equations **B D E F M**

10. [*10 points*] State one common feature of the work of Newton and Leibniz on calculus that went beyond previous results, and state one way in which the approaches or results of Newton and Leibniz were different from each other.

SOLUTION

Advances of Newton and Leibniz.

Development of general processes
Applying the methods to more general classes of problems
Emphasis on the inverse relation between differentiation and integration
Formulation of results more uniformly in algebraic terms

Unique features of Newton's approach.

Development of the infinite binomial series for $(1 + x)^a$
Emphasis on problems involving motion
Priority placed on differentiation as opposed to integration
Emphasis on expressing functions with infinite series
Defined the sorts of problems considered by successors

Unique features of Leibniz' approach.

More systematic use of infinitesimals
Emphasis on maximum/minimum problems and differential equations
Emphasis on integration as opposed to differentiation
Emphasis on expressing functions by finite expressions involving standard operations
Defined the way the subject is formulated and its basic notation