6. Mathematics of Asian and Arabic civilizations — I

(Burton, **5.3, 5.5, 6.1**)

Every civilization needs to develop some mathematical knowledge in order to succeed, and several other ancient civilizations went quite far in producing substantial amounts of mathematics. Not surprisingly, many fundamental mathematical ideas were discovered independently in each of these civilizations, but there were also noteworthy differences in the organization and emphasis of the subject, and in many cases one civilization discovered things which others did not. We shall begin this unit with brief discussions of mathematics in the civilizations of China and India. As indicated at the beginning of this course, we shall not try to do this comprehensively, but instead we shall try to focus on relatively unique features of mathematics in these civilizations and on advances which influenced the development of the mathematics we have in our contemporary civilization. We shall pay particular attention to the contributions of Indian civilization in these notes because the work of Indian mathematicians has had a particularly strong impact on mathematics as we know it today.

Mathematical activity in China

It seems clear that significant Chinese work in mathematics goes back at least 3000 years and probably at least a millennium longer, but our knowledge prior to about 100 B.C.E. is sketchy and often quite speculative. However, by that time the Chinese mathematics was already quite well developed. The level of Chinese mathematical knowledge and ability at the time can be seen from the contents of ***The Nine Chapters on the Mathematical Art***, which was apparently written in the 1st century B.C.E. and was extremely influential.

One important difference between Chinese and Greek mathematics involved the role of logic. Chinese mathematics did not use deductive logic as the framework for the subject, and the interest was more directed towards solving wide ranges of problems and obtaining numerically accurate solutions than to studying the subject for its own sake or describing solutions qualitatively. The Chinese algorithm for extracting square roots, which was widely taught in American schools during the first few decades of the 20th century, illustrates these features of Chinese mathematics. Here are online references for the algorithm and its justification:

http://www.homeschoolmath.net/teaching/square-root-algorithm.php

http://www.homeschoolmath.net/teaching/sqr-algorithm-why-works.php

However, there was also interest in some topics which were not intrinsically practical. For example, the Chinese were fond of patterns, and at some very early point they discovered the existence of magic squares, which are square matrices of positive integers such that the sums of the rows, columns and diagonals are all equal to some fixed value. Further discussions of this topic are in the following two online files:

http://math.ucr.edu/~res/math153/oldmagicsquare.pdf

http://math.ucr.edu/~res/math153/oldmagicsquare2.pdf

Other noteworthy features of early Chinese mathematics include relatively advanced methods for approximate numerical solutions (versions of the procedure called ***Horner's method***, named after W. G. Horner, 1786 – 1837; for a description see **http://mathworld.wolfram.com/HornersMethod.html**) and systematic procedures for solving systems of linear equations and the following basic class of problems the solutions are given by a result known as the ***Chinese Remainder Theorem***:

Suppose that we are given two relatively prime positive integers p and q, and let n be a third positive integer. Suppose further that long division of n by p and q yields remainders of a and b respectively. What value(s) must n take?

For example, if $p = 3$ and $q = 5$, and the respective remainders are 2 and 3 respectively, then n must have the form $8 + 15m$ for some integer m. The file **http://math.ucr.edu/~res/math153/chineseremainder.pdf** solves this and other problems of the same sort.

Before moving ahead, we shall give another reference for Horner's method; namely, pages 174 – 177 from the following old college algebra textbook:

> A. A. Albert. ***College Algebra*** (Reprint of the 1946 Edition). University of Chicago Press, Chicago IL, 1963.

The peak period of Chinese mathematics took place during the 13$^{\text{th}}$ century and early 14$^{\text{th}}$ century. One of the most prominent figures of the time was Qin Jushao (1202 – 1261), whose ***Mathematical Treatise in Nine Sections*** covers many of the topics discussed before at more sophisticated levels, including polynomial equations with degrees up to 10 and some equations of Diophantine type which went further than the problems which Diophantus is known to have considered. Solutions to relatively complicated systems of equations were also central points in the research of other major figures from that era including Li Zhi (also called Li Yeh, 1192 – 1279), Yang Hui ($c.$ 1238 – $c.$1298) and Zhu Shijie (1260 – 1320). Yang Hui also wrote an extensive works on magic squares and mathematical education.

Although Chinese mathematical activity declined after the peak period, it never really stopped, and in fact elements of the Chinese mathematical tradition continued even after the infusion of mathematical knowledge from the West which began with the missionary work of M. Ricci (1552 – 1610).

Mathematical activity in India

Indian mathematics has a long and interesting history, probably going back at least 4000 years with a sequence of distinct eras during the ancient period (the **Harappan** or **Indus Valley** era until about 1500 B.C.E, the **Vedic** era from about 1500 B.C.E. until about 500 B.C.E., and the **Jaina** era from about 500 B.C.E. until about 500 A.D.). Although there may have been some mathematical interactions between Indian mathematics and Greek or Chinese mathematics, it is also clear that the Indian approach to the subject contained concepts and ideas that were not well developed by either of the other civilizations. In keeping with the focus of this course, we shall begin our discussion of Indian mathematics with comments on its distinctive features and its advances which ultimately had a major impact on modern mathematics.

Logical structure played a more significant role in Indian mathematics than in Chinese mathematics, but it was definitely not comparable to the place of logic in Greek mathematics. Another noteworthy difference between Indian and Greek mathematics is that Indian mathematicians were less troubled by distinctions between rational and irrational numbers, and in fact they were far more willing to consider still other concepts including negative numbers, zero, and even infinite objects in some cases. Ancient Indian mathematics is also distinguished by its extensive use of poetic language in its mathematical writings; one apparent reason for this was relative ease of memorization. In a similar vein, the mathematical problems in Indian mathematical writings often were placed into highly imaginative settings.

Although studies of grammatical structure have only recently become linked to the mathematical sciences, linguists like N. Chomsky (1928 –), computer scientists like J. Backus (the developer of **FORTRAN**, 1924 – 2007) and many others have forcefully demonstrated the connection between the two subjects and their importance for each other. In view of this, the extensive work of Pāṇini (*c.* 520 – *c.* 460 B.C.E.) on Sanskrit grammar definitely deserves to be included as a contribution to mathematics as we know it today; his studies strongly anticipated much of the 20^{th} century work on the grammatical structures that is fundamentally important for operating computers (and one important concept is known as the ***Panini – Backus normal form***). Somewhat later writings of Pingala (probably between 400 and 100 B.C.E.) on prosody (PROSS – o – dee, the rhythm, stress, and intonation of language, usually in its spoken form) contains the first known description of a binary numeral system. The study of language patterns led to substantial work on combinatorial (counting) problems, and in several respects Pingala's results appear to have anticipated important later developments.

Probably the best known, and widely used, legacy of Indian mathematics is the base **10** place value numeration system that we use today, with nine basic digits arranged in sequences and the roles of the digits determined by their placement. With the ultimate incorporation of zero into the framework, the nine digit system grew to the ten digit notation that has become standard worldwide. The impact of this discovery is stated clearly and concisely in the following quotation from P. – S. Laplace (1749 – 1847):

> The ingenious method of expressing every possible number using a set of ten symbols (each symbol having a place value and an absolute value) emerged in India. The idea seems so simple nowadays that its significance and profound importance is no longer appreciated. Its simplicity lies in the way it facilitated calculation and placed arithmetic foremost amongst useful inventions. The importance of this invention is more readily appreciated when one considers that it was beyond the two greatest men of Antiquity, Archimedes and Apollonius.

Some comparisons with numeration in other civilizations may be enlightening. The Babylonians actually had a base **60** place value numeration system, but the Greek numeration system did not, and in fact it was more similar in structure to Roman numerals, where a number like **234** was written using a special symbol combination for **60 (CC),** another special symbol combination for **30 (XXX),** and yet another special symbol combination for **4 (IV).** However, the Greek conventions only involved single symbols for the hunrdeds, tens and units terms, using their alphabet at the time for the symbols. Here is a chart giving the symbols for the various numbers. Three symbols in this table correspond to letters which are no longer used in the language; namely, Ϝ (***wau*** or ***digamma***) or ϛ (***stigma***) for **6,** ϟ or ϙ (***qoppa***) for **90,** and ϡ (***sampi***) for **900.**

Letter	Value	Letter	Value	Letter	Value
α	1	ι	10	ρ	100
β	2	κ	20	σ	200
γ	3	λ	30	τ	300
δ	4	μ	40	υ	400
ε	5	ν	50	φ	500
Ϝ or ϛ	6	ξ	60	χ	600
ζ	7	ο	70	ψ	700
η	8	π	80	ω	800
θ	9	ϟ or ϙ	90	ϡ	900

(Source: http://en.wikipedia.org/wiki/Greek_numerals)

For example, the traditional representation of the celebrated number **666** in Greek versions of the New Testament was χξϛʹ; four digit numbers were formed by using the symbols for units in the thousands place with a special accent sign — to illustrate this, we note that the number **2012** would have been written as **͵βιβʹ.**

Historians have varying opinions on exactly when the Indian place value system was developed ranging from the 1st to 5th century A.D., probably with most in the 2nd or 3rd century A.D., but ancient Indian mathematics has very little to say about exactly when discoveries were made or who made them. Since the idea of using nine or ten digits appears explicitly as a well – known technique in the writings of Āryabhaṭa the Elder (476 – 550) and he is regarded as the earliest mathematical contributor to the classical era of Indian mathematics from about 500 to 1200 A.D., we shall begin with his work.

The surviving mathematical work of Aryabhata is contained in a manuscript called the ***Aryabhatiya***, which is written entirely in verse and also covers other subjects besides mathematics. As noted before, there is a passing reference describing a numbering system like the one we use today, and the mathematical portion of the work also contains results on integral solutions to Diophantine equations of the first and second degree. Trigonometry also played a significant role in Indian mathematics, and in fact modern trigonometry follows the Indian approach — which is based upon the sine function — rather than the Greek approach, which was based upon the chord function **crd** discussed previously. The tables in the ***Aryabhatiya*** define trigonometric functions for angles with a basic increment of **3.75** degrees.

One of the most important figures in Indian mathematics was Brahmagupta (598 – 670), whose writings contain many important and far – reaching ideas. We shall list a few of them:

1. He explicitly recognized that Diophantine equations can have many solutions.

2. He used nine or ten symbols to write numbers (Aryabhata used an older alphabetic system).

3. He had no reservations about working with negative numbers and irrationals.

4. His work recognizes the concept of zero, although the first known explicit use of a symbol for zero in written Indian mathematics does not occur until late in the 9th century.

5. He devoted a great deal of effort to analyzing Diophantine equations like the previously discussed Pell equation $x^2 = 1 + ay^2$. Further results on this equation due to Bhāskara (1114 – 1185) are mentioned below; Brahmagupta's main contribution was to give a method for constructing new solutions out of previously known ones (a method for doing so is given in **http://math.ucr.edu/~res/math153/history06b.pdf**).

Brahmagupta's writings also treat geometrical topics, but some of his conclusions are extremely inaccurate and far below the quality of his algebraic results. However, one particularly noteworthy geometric result due to him is an area formula for a quadrilateral that can be inscribed in a circle (see Exercise **6** on page 193 of Burton). Further information on proofs for this result may be found at the following online sites:

http://jwilson.coe.uga.edu/emt725/brahmagupta/brahmagupta.html

http://en.wikipedia.org/wiki/Brahmagupta's_formula

The work of Mahāvīra (or Mahaviracharya, *c*. 800 – *c*. 870) clarified and extended the ideas of Aryabhata and Brahmagupta, and his only surviving work is the earliest known Indian text which is devoted entirely to mathematics. Like other mathematicians from the classical era, he discussed arithmetic operations involving zero, and like the others he found the concept of division by zero to be troublesome. Brahmagupta had tried to explain division by zero, but he did not get very far and mistakenly asserted that $0/0$ should equal 0. Mahavira mistakenly suggested that division by zero had no effect on the number being divided. More than two centuries would pass until Bhāskara, the next epic figure in the history of Indian mathematics, suggested that "if one divides a finite nonzero number by zero, the result is infinity," which is now regarded as one of the best possible descriptions (however, this statement can lead to paradoxes if it is not used carefully). For the sake of completeness, here are some mathematically accurate online references concerning division by zero:

http://mathworld.wolfram.com/DivisionbyZero.html

http://mathworld.wolfram.com/AffinelyExtendedRealNumbers.html

http://mathworld.wolfram.com/ProjectivelyExtendedRealNumbers.html

http://en.wikipedia.org/wiki/Extended_real_number_line

http://en.wikipedia.org/wiki/Projectively_extended_real_numbers

We turn now to Bhāskara (also known as Bhaskara II or Bhaskarachariya), who was also one of the most important figures in Indian mathematics. The concept of zero is far more explicit in his work, and as noted above he made a crucial advance in our efforts to understand the issues involving division by zero. Also, he clearly understood that quadratic equations have two roots; one verbal problem in his writings yields the equation $x^2 = 64(x - 12)$, and he notes that 16 and 48 are both valid solutions. His mathematical writings also used the decimal system methodically to an unprecedented degree.

Some of Bhaskara's deepest discoveries involve Pell's equation $px^2 + 1 = y^2$. His numerical results on this equation include the following: For $p = 61$ there is a solution $x = 226153980$, $y = 1776319049$, and for $p = 67$ there is a solution $x = 5967$, $y = 48842$. More generally, Bhaskara developed a very elegant algorithmic "cyclic method" for finding a minimal solution to Pell's equation when p has no square divisors except 1; A description of this method appears on pages 223 – 225 of the book by Katz or pages 10 – 11 of the following online document:

http://www.math.ucla.edu/~vsv/gamelin.pdf

Although Bhaskara's method is relatively simple and efficient, a proof that it always works was not given until 1929 in the following paper:

> **K. A. A. Ayyangar, *New light on Bhaskara's Chakravala or cyclic method of solving indeterminate equations of the second degree in two variables.*** Journal of the Indian Mathematics Society **18** (1929), 232 – 245.

The following paper contains still further information:

> **C. – O. Selenius. *Rationale of the chakravāla process of Jayadeva and Bhāskara II.*** Historia Mathematica **2** (1975), 167 – 184.

Bhaskara also made noteworthy contributions in other areas. For example, in his work on astronomy he broke new ground in studying trigonometry for its own sake rather than for its computational value, and he had many insights which were rudimentary versions of key facts in differential and integral calculus.

The classical period in Indian mathematics essentially basically ended with Bhaskara; a series of outside invasions cause major disruptions and changes, especially in the northern part of India (the late 12th century saw the founding of the Sultanate of Delhi; see http://www.infoplease.com/ce6/history/A0815061.html). However, some mathematical activity continued in the south, particularly in Kerala, which is at the southwest tip of the Indian peninsula. The Kerala school of mathematics built upon the work of Bhaskara on astronomy and trigonometry, and the most renowned achievements were strong results on infinite series related to trigonometric functions. For example, Madhara of Sangamagrama (1340 – 1425) discovered the standard infinite series for $\arctan x$, and subsequently Nilakantha Somayaji (1444 – 1544) found an infinite series for $\pi/4 = \arctan 1$ that converges much more rapidly than the standard series for $\arctan 1$ (more details and some references are given in the next unit). In many respects the results of the Kerala school foreshadowed the development of calculus (although many key ingredients in calculus were missing and claims that their discoveries were transmitted to the Western world are not supported by direct evidence; however, it is conceivable that Western missionaries or merchants might have passed along information about the findings of the Kerala school). Aside from the Kerala school's results on infinite series and trigonometric functions, an early version of the Mean Value Theorem due was obtained by Parameshvara (1370–1460).

The original discoveries of the Kerala school appear to have dried up in the early years of the 17th century, but the school itself survived for about another two centuries, with the final activity a few decades before the British colonial era. Shortly afterwards, Western mathematics began to exert a strong influence on the Indian subcontinent. However, mathematics in India retained some traditional features at least for a while, and some historians have cited the extraordinary mathematical studies of S. A. Rāmānujan (1887 – 1920) as an example of this phenomenon. More information about this unique mathematician and his work can be found in the following online references:

http://en.wikipedia.org/wiki/Srinivasa_Ramanujan

http://www.usna.edu/Users/math/meh/ramanujan.html

http://scienceworld.wolfram.com/biography/Ramanujan.html

Since the most far – reaching consequences for modern mathematics were transmitted to the Western World indirectly through Arabic/Islamic civilization, we shall move on to the latter after a few final remarks in this half of the unit.

Remarks on other non – Western cultures

We shall mention a few particularly striking achievements in other cultures.

Given that it took many civilizations a long time to recognize the concept of zero and to use it in their numbering systems, it is remarkable that zero was included in Mayan and pre – Mayan numeration systems more than 2000 years ago. The Mayans had a very complicated and well – developed calendar which used the zero concept, and some objects have Mayan calendar dates corresponding to approximately 30 B.C.E..

Some features of Japanese mathematics also deserve to be mentioned. Near the beginning of the 17th century, a sophisticated and highly original school of mathematics developed while the country was isolated from the outside world. This work was largely based upon Chinese mathematics, but it went far beyond the latter in some respects and had its own style. One of the most celebrated mathematicians of this school was Takakazu (or Kōwa) Seki (1642 – 1708). He did a considerable amount of work on infinitesimal calculus and Diophantine equations, and he also made important discoveries related to matrix algebra (more precisely, the theory of determinants). Seki was a contemporary of Gottfried Leibniz and Isaac Newton, but his work was entirely independent. Some important aspects of Seki's work were carried further by Wada Yenzō Nei (1787 – 1840). With the opening of Japan to the outside world in the middle of the 19th century, the influence of Western mathematics led to the rapid decline and assimilation of the Japanese school. The following book is (still) a good source of information about this aspect of mathematical history:

D. E. Smith and Y. Mikami. ***A History of Japanese Mathematics***
(Reprint of the 1914 edition). Dover Publications, New York, 2004.