7.A. The closed formula for Fibonacci numbers

We shall give a derivation of the closed formula for F_n here. This formula is often known as ***Binet's formula*** because it was derived and published by J. Binet (1786 – 1856) in 1843. However, the same result had been known to several prominent mathematicians — including L. Euler (1707 – 1783), D. Bernoulli (1700 – 1782) and A. De Moivre (1667 – 1754) — more than a century earlier.

We begin with general observations. Suppose we are given a sequence x_n which is defined recursively by a formula

$$x_n = bx_{n-1} + cx_{n-2}$$

where b and c are real numbers. The first point is that such a sequence is uniquely determined by x_0 and x_1. Suppose that x_n and y_n are two such sequences with values of P and Q when $n = 0$ or 1. Then $x_n = y_n$ when $n = 0$ or 1; assume that the values of the sequence are equal for all $n < k$, where $k > 1$. Then we have

$$x_k = bx_{k-1} + cx_{k-2} = by_{k-1} + cy_{k-2} = y_k$$

and therefore the two sequences are equal by mathematical induction.

In favorable cases one can write down the sequence x_n in a simple and explicit form. Here is the key step which also applies to a wide range of similar problems.

<u>**PROPOSITION.**</u> *Suppose that* r *and* s *are distinct roots of the* <u>***auxiliary polynomial***</u> $g(t) = t^2 - bt - c$. *Then for every pair of constants* u, v *the sequence* $ur^n + vs^n$ *solves the* <u>***finite difference equation***</u> $x_n = bx_{n-1} + cx_{n-2}$.

<u>***Derivation.***</u> Let $y_n = ur^n + vs^n$; we need to show that

$$y_n - by_{n-1} - cy_{n-2} = 0$$

for all $n > 1$. If we expand the left hand side and we obtain the following equations.

$$(ur^n + vs^n) - b(ur^{n-1} + vs^{n-1}) - c(ur^{n-2} + vs^{n-2}) =$$

$$u(r^n - br^{n-1} - cr^{n-2}) + v(s^n - bs^{n-1} - cs^{n-2}) =$$

$$ur^{n-2}g(r) + vs^{n-2}g(s) = ur^{n-2}\cdot 0 + vs^{n-2}\cdot 0 = 0.$$

Therefore $x_n = ur^n + vs^n$ solves the original equation.

One can take this further to find the unique solutions satisfying $x_0 = P$ and $x_1 = Q$ by solving the equations $u + v = P$ and $ur + vs = Q$ for u and v. It is always possible to find a unique solution because r and s are distinct.■

We shall now apply all this to the Fibonacci equation

$$F_n = F_{n-1} + F_{n-2}$$

whose auxiliary polynomial is equal to

$$x^2 = x + 1.$$

Equivalently, one can write this polynomial in the form

$$x^2 - x - 1 = 0$$

and since the roots of this equation are

$$\phi = \frac{1+\sqrt{5}}{2}, \qquad \psi = \frac{1-\sqrt{5}}{2}$$

it follow that the closed formula for the Fibonacci sequence must be of the form

$$f_n = u\phi^n + v\psi^n$$

for some constants u and v. If we now use the conditions $F_0 = 0$ and $F_1 = 1$, we see that

$$0 = u\phi^0 + v\psi^0, \qquad 1 = u\phi^1 + v\psi^1$$

where the first equation simplifies to $u = -v$; substituting this into the second one yields

$$1 = u\left(\frac{1+\sqrt{5}}{2}\right) - u\left(\frac{1-\sqrt{5}}{2}\right) = u\left(\frac{2\sqrt{5}}{2}\right) = u\sqrt{5}.$$

Therefore

$$u = \frac{1}{\sqrt{5}}, \qquad v = \frac{-1}{\sqrt{5}}$$

and accordingly we have

$$f_n = \frac{\phi^n}{\sqrt{5}} - \frac{\psi^n}{\sqrt{5}} = \frac{\phi^n - \psi^n}{\sqrt{5}}. \blacksquare$$