12. The development of calculus

(Burton, **8.3 – 8.4**)

We have already noted that certain ancient Greek mathematicians — most notably Eudoxus and Archimedes — had successfully studied some of the basic problems and ideas from integral calculus, and we noted that their method of exhaustion was similar to the modern approach in some respects (successive geometric approximations using figures whose measurements were already known) and different in others (there was no limit concept, and instead there were delicate *reductio ad absurdum* arguments). Towards the end of the 16th century there was further interest in the sorts of problems that Archimedes studied, and later in the 17th century this led to the independent development of calculus in Europe by Isaac Newton (1643 – 1727) and Gottfried Wilhelm von Leibniz (1646 – 1716).

Two important factors led to the development of calculus originated in the late Middle Ages, and both represented attempts to move beyond the bounds of ancient Greek mathematics, philosophy and physics.

1. *Interest in questions about infinite processes and objects.* As noted earlier, ancient Greek mathematics was not equipped to deal effectively with Zeno's paradoxes about infinite and by the time of Aristotle it had essentially insulated itself from its "horror of the infinite." However, we have also noted that Indian mathematicians did not share this reluctance to work with the concept of infinity, and Chinese mathematicians also used methods like infinite series when these were convenient. During the 14th century the mathematical work of Oresme and others on infinite series complemented the interests of the scholastic philosophers; questions about the infinite played a key role in the efforts of scholastic philosophy to integrate Greek philosophy with Christianity. Philosophical writings by scholars including William of Ockham (1285 – 1349), Gregory of Rimini (1300 – 1358), and Nicholas of Cusa (1401 – 1464) provided insights which anticipated the concept of limit — a basic part of calculus as we know it.

2. *Interest in questions about physical motion.* During the 13th century Jordanus de Nemore discovered the mathematically correct description for the physics of an inclined plane (a result that had eluded Archimedes and was described incorrectly by Pappus), and later members of the Merton school in Oxford such as T. Bradwardine (1290 – 1349) and W. Heytesbury (1313 – 1373) had discovered an important property of uniformly accelerated motion; namely, the ***average velocity*** (total distance divided by total time) is in fact the mathematical average of the initial and final velocities. The significance of this concept for the motion of freely falling bodies was not understood at the time and would not be known until the work of Galileo and others in the 16th century. Oresme's writings on graphs contain an early insight that the area under a graph of

velocity with respect to time is equal to the total distance traveled. During the late 16th century interaction between mathematics and physics began to increase at a much faster rate, and many important contributors to mathematics such as S. Stevin also made important contributions to physics (in Stevin's case, the centers of mass for certain objects and the statics and dynamics of fluids).

Here are some online references to further information about the topics from the Middle Ages described above:

http://www.math.tamu.edu/%7Edallen/masters/medieval/medieval.pdf

http://www.math.tamu.edu/%7Edallen/masters/infinity/infinity.pdf

http://plato.stanford.edu/entries/heytesbury/

Problems leading to the development of calculus

In the early 17th century mathematicians became interested in several types of problems, partly because these were motivated by advances in physics and partly because they were viewed as interesting in their own right.

1. Measurement questions such as lengths of curves, areas of planar regions and surfaces, and volumes of solid regions, and also finding the centers of mass for such objects.
2. Geometric and physical attributes of curves, such as tangents and normals to curves, the concept of curvature, and the relation of these to questions about velocity and acceleration.
3. Maximum and minimum principles; ***e.g.***, the maximum height achieved by a projectile in motion or the greatest area enclosed by a rectangle with a given perimeter.

Specific examples of all three types of problems had already been studied by Greek mathematicians. The results of Eudoxus and Archimedes on the first type or problem and of Archimedes and Apollonius on the second have already been discussed. We know about the work of Zenodorus (200 – 140 B.C.E.) on problems in the third type because it is reproduced in Pappus' anthology of mathematical works (the previously cited ***Collection*** or ***Synagoge***). Some examples of Zenodorus' results are

(1) among all regular n – gons with a fixed perimeter, a regular n – gon encloses the greatest area,
(2) given a polygon and a circle such that the perimeter of the polygon equals the circumference of the circle, the latter encloses the greater area,
(3) a sphere encloses the greatest volume of all surfaces with a fixed surface area.

GIVE REFERENCES

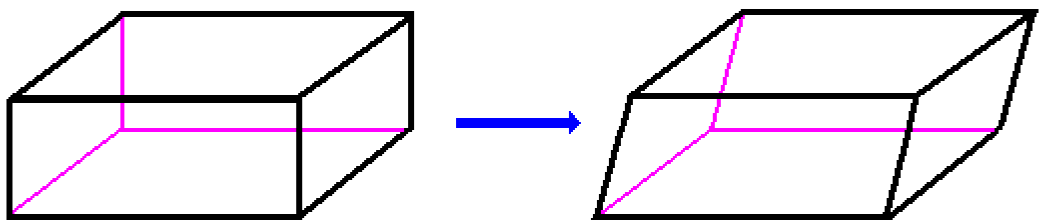
Infinitesimals and Cavalieri's Principle

We shall attempt to illustrate the concept of infinitesimals with an example of its use to determine the volume of a geometrical figure.

Usually the formula for the volume of a cone is derived only for a ***right circular cone*** in which the axis line through the vertex or nappe is perpendicular to the plane of the base. It is naturally to ask whether there is a similar formula for the volume of an oblique cone for which the axis line is not perpendicular to the plane of the base (see the drawing below). One can answer this question using an approach developed by B. Cavalieri (1598 – 1647). Similar ideas had been considered by Zu Chongzhi, also known as Tsu Ch'ung – chin (429 – 501), but the latter's work was not known to Europeans at the time.

<u>CAVALIERI'S PRINCIPLE.</u> *Suppose that we have a pair three-dimensional solids* **S** *and* **T** *that lie between two parallel planes* $\mathbf{P_1}$ *and* $\mathbf{P_2}$, *and suppose further that for each plane* **Q** *that is parallel to the latter and between them the plane sections* $\mathbf{Q} \cap \mathbf{S}$ *and* $\mathbf{Q} \cap \mathbf{T}$ *have equal areas. Then the volumes of* **S** *and* **T** *are equal.*

Here is a physical demonstration which suggests this result: Take two identical decks of cards that are neatly stacked, just as they come right out of the package. Leave one untouched, and for the second deck push along one of the vertical edges so that the deck forms a rectangular parallelepiped as below.



In this new configuration the second deck has the same volume as first and it is built out of very thin rectangular pieces (the individual cards) whose areas are the same as those of the corresponding cards in the first deck. So the areas of the plane sections given by the separate cards are the same and the volumes of the solids formed from the decks are also equal.

We shall now apply this principle to cones. Suppose we have an oblique cone as on the right hand side of the figure below. On the left hand side suppose we have a right circular cone with the same height and a circular base whose area is equal to that of the elliptical base for the second cone.

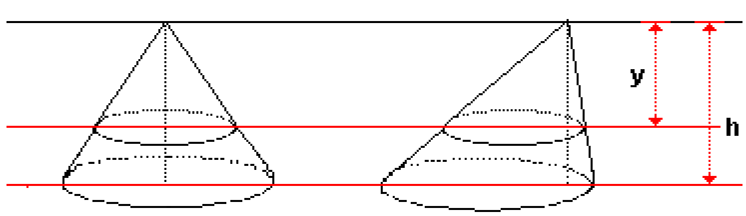


Figure illustrating Cavalieri's Principle

In the notation of Cavalieri's Principle, we can take $\mathbf{P_1}$ to be the plane containing the bases of the two cones and $\mathbf{P_2}$ to be the plane which contains their vertices and is parallel to $\mathbf{P_1}$. Let **Q** be a plane that is parallel to both of them such that the constant distance between $\mathbf{P_2}$ and **Q** is equal to y; we shall let h denote the distance between $\mathbf{P_1}$ and $\mathbf{P_2}$, so that h is also the altitude of both cones. If b denotes the areas of the bases of both cones then the areas of the sections formed by intersecting these cones with **Q** are both equal to by^2/h^2. Therefore Cavalieri's Principle implies that both cones have the same volume, and since the volume of the right circular cone is $bh^2/3$ it follows that the same is true for the oblique cone.

One approach to phrasing our physical motivation in mathematical terms is to imagine each cone as being a union of a family of solid regions given by the plane sections; for the right circular cone these are regions bounded by circles, and for the oblique cone they are bounded by ellipses. Suppose we think of these sections as representing cylinders that are extremely thin. Then in each case one can imagine that the volume is formed by adding together the volumes of these cylinders, whose areas — and presumably thicknesses — are all the same, and of course this implies that the volumes of the original figures are the same. One fundamental question in this approach is to be more specific about the meaning of "extremely thin." Since a planar figure has no finite thickness, one might imagine that the thickness is something less, and this is how one is led to the concept of a thickness that is ***infinitesimally small.***

The preceding discussion suggests that an infinitesimal quantity is supposed to be nonzero but is in some sense smaller than any finite quantity. If we are given a geometric figure that can be viewed as a union of "indivisible" objects with one less dimension — for example, the planar region bounded by a rectangle viewed as a union of line segments parallel to two of the sides, or the solid region bounded by a cube as a union of planar regions bounded by squares parallel to two of the faces — then the idea is to view the square as a union of rectangular regions with infinitesimally small width or the square as a union of solid rectangular regions with infinitesimally small height. Likewise, this approach suggests an interpretation of a continuous curve as being composed of a family of straight lines with infinitesimally small length. When applied to our examples of cones, it leads to thinking of the solid region bounded by either cone as a union of circular or elliptical cylinders with infinitesimally small height.

Mathematicians and users of mathematics have though about infinitesimals for a long time. They already appear in the mathematics of the early Greek atomist philosopher Democritus (460 – 370 B.C.E.) and an approach to squaring the circle developed by Antiphon, but advances by Eudoxus and others during the 4th century B.C.E. enabled Greek mathematicians to avoid the concept, and this fit perfectly with the reluctance of Greek mathematicians and philosophers (*e.g.*, Aristotle) to eliminate questions about the infinite from their mathematics. Taking the somewhat obscure form of "indivisibles," they reappeared in the mathematics of the late Middle Ages, and they played an important role in the work of J. Kepler (1571 – 1630) on laws of planetary motion, particularly his Second Law which states that the orbits of planets around the sun sweep out equal areas over equal times. During the 17th century infinitesimals were used freely by many mathematicians and scientists who contributed to the development of calculus, and in particular both Newton and Leibniz used the concept in their definitive accounts of the subject. However, as calculus continued to develop, doubts about the logical soundness of infinitesimals also began to mount. Such questions ultimately had very important consequences for the development of mathematics, and they will be discussed later in the next unit.

A brief discussion of Cavalieri's Theorem, its importance for classical geometry, and its interpretation in terms of integral calculus appears on pages 156 – 159 (= document pages 15 – 17) of the following online reference:

http://math.ucr.edu/~res/math133/geometrynotes3c.pdf

A general comment on coverage and attribution

During the 17th century many mathematicians were interested in similar problems, and many results were discovered independently by two or more researchers. Not all such cases can be described completely in a brief summary such as these notes; one guiding principle here is to mention the persons whose work on a given problem had the most impact. The following quotation from pages 306 – 307 of Eves' ***Introduction to the History of Mathematics*** (6th Edition) summarizes the basic viewpoint of these notes:

> It is only fair to note here two facts that will contribute to the somewhat unbalanced presentation of the history of mathematics in the … [remaining] part … The first of these is that mathematical activity began to grow at so great a rate that henceforth many names must be omitted that might have been considered in a less productive period. The second fact is that, with the unfolding of the 17th century, an increasing amount of mathematical research occurred that cannot be appreciated by a general reader [who has not taken courses beyond first year calculus and discrete mathematics], for it has been rightfully claimed that the history of a subject [in this case, mathematics beyond the previously cited courses] cannot be properly understood without a knowledge of the subject itself.

Progress on measurement questions

Methods and results from Archimedes and others provided important background and motivation for work in this area. We have already discussed the use of infinitesimals to derive formulas in some cases, and there was a great deal of further work based upon such ideas. In particular, during the time before the appearance of Newton's and Leibniz' work, many of the standard examples in integral calculus had been worked out by preliminary versions of methods that became standard parts of the subject. Here are some specific examples:

Integrals of polynomials and more general power functions. Cavalieri computed the integral of x^n geometrically in cases where n is a positive integer, Gregory of St. Vincent (1584 – 1667) integrated $1/x$ in geometric terms that are equivalent to the usual formula involving $\log_e x$, and J. Wallis (1616 – 1703) generalized the integral formula for x^n to other real values of n. — Wallis was a particularly important figure in the development of calculus for several reasons. His methods, which are discussed on pages 357 – 360 of Burton, replaced geometric techniques with algebraic computations and analytic considerations, and as such they are a milestone in the development of analysis (calculus) as a subject distinct from both algebra and geometry. In an entirely different direction, Wallis is also known today for his applications of integral formulas to derive his infinite product formula for $\pi/2$:

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{8}{7}\cdot\frac{8}{9}\cdots$$

We should note here that an earlier infinite product formula involving π

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2+\sqrt{2}}}{2}\cdot\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots$$

had been discovered by Viète (and, as noted earlier, Indian mathematicians had already discovered some of the standard infinite series formulas involving π). **REFERENCES**

<u>Measurements involving the cycloid curve and regions partially bounded by it.</u> There was an enormous amount of interest in the properties of the cycloid curve during the 17th century, and there were also some bitter disputes about priorities among some of the numerous mathematicians who worked on this example. Results included computations of arc lengths as well as areas, volumes and centers of mass associated to the curve. Mathematicians whose names are associated with this work include B. Pascal (1623 – 1662), E. Torricelli (1608 – 1647), and G. P. de Roberval (1602 – 1675).

<u>Infinite solids with finite volume.</u> Torricelli also discovered a fact that few if any mathematicians had anticipated; namely, the existence of an unbounded solid of revolution whose volume is finite. His example is an unbounded piece of the solid formed by rotating the standard hyperbola $y = 1/x$ about the $x-$ axis.

<u>Arc length.</u> Early in the 17th century there were doubts about the possibility of computing the arc lengths of many curves, including some extremely familiar examples. Results of H. van Heuraet (1633 – 1660) showed the problem of finding arc length of a given curve is equivalent to determining the area under another curve, and he also worked out certain examples including the semi – cubical parabola $ay^2 = x^3$. The arc length of a spiral curve was computed by Roberval.

<u>Integrals and series expansions of transcendental functions.</u> Results on the integrals of the standard trigonometric functions were obtained by Pascal, Roberval, I. Barrow (1630 – 1677) and J. Gregory (1638 – 1675). The standard infinite series expansion for $\arctan x$ was obtained by Gregory (however, as noted before, the Indian mathematician Madhava had discovered the formula three centuries earlier), and the standard infinite series expansion for $\log_e(1+x)$ was obtained by N. Mercator (1620 – 1687); the latter should not be confused with the mathematical cartographer G. Mercator mentioned earlier. Gregory also made numerous other contributions, including extending and applying the classical method of exhaustion to questions about other conic sections and writing the first text covering the material that would become calculus.

Progress in differentiation and maximization/minimization problems

Greek mathematics provided far less insight into questions about tangent lines or maximizing functions than it did for computing areas and volumes. In particular, there were no general principles comparable to the method of exhaustion for describing tangents; each example was treated in an entirely separate manner. Therefore one major problem facing 17th century mathematicians was to produce a workable concept of tangent line.

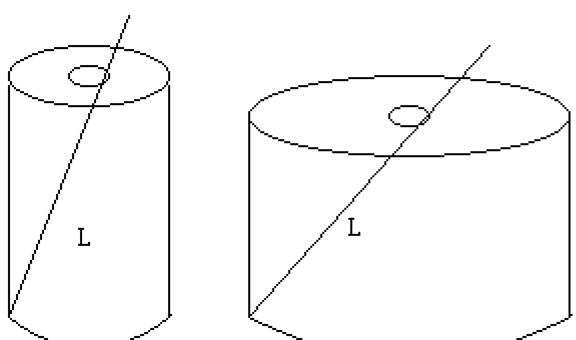
Some names particularly associated with this problem are Descartes, Fermat, Roberval and Barrow. Descartes' approach was based on finding the normal (perpendicular) line to a curve at a point, and Roberval's was motivated by the standard interpretation of a parametrized curve as the path of a moving object. Fermat and Barrow both defined tangent lines by the method that became standard; namely, the tangent line to a curve **C** at a point x is a limit of the secant lines joining x to a second point y on **C** as y approaches x. Fermat understood the basic idea, but Barrow's language (***i.e.***, his ***differential triangle***) was more precise. A discussion of Barrow's work appears on pages 388 – 391 of Burton.

As in the case of measurement problems, examples were the focal point of work on tangents. The standard results on the slopes of tangent lines for polynomial graphs were obtained by Fermat in the monomial case and in general by Hudde. Applications of derivatives to repeated polynomial roots were also discovered at this time. Tangent lines to the cycloid were determined independently by Descartes, Fermat and Roberval.

Fermat also studied maximization and minimization problems using the approach he developed for tangent lines. We note that he was interested in several different types of minimization problems in mathematics and physics, including the determination of a point inside a triangle such that the sum of the distances to all three sides is minimized, and more significantly his Least Time Principle in optics, which states that a beam of light will take the path from one point to another that takes the shortest amount of time and yields the standard physical laws governing refraction and reflection. Further discussion of such basic minimization problems is contained in the following book:

S. Hildebrandt and A. Tromba. ***The Parsimonious Universe*.**
Springer – Verlag, New York, 1996. **ISBN:** 0 – 387 – 97991 – 3.

A result of Kepler's on maximization (the ***Wine Barrel Problem***) can also be mentioned at this point. He observed that a wine merchant had figured the amount of wine in a barrel by inserting a rod into the barrel diagonally through a small hole in the top. When the rod was removed, the length L of the rod which was wet determined the price for the barrel. Kepler was concerned about the uniformity of pricing by this method and decided to analyze its accuracy; he correctly realized that a taller, narrower barrel might yield the same measurement as a shorter, wider one, resulting in the same wine price, even though its volume would be considerably smaller (see the figure below).

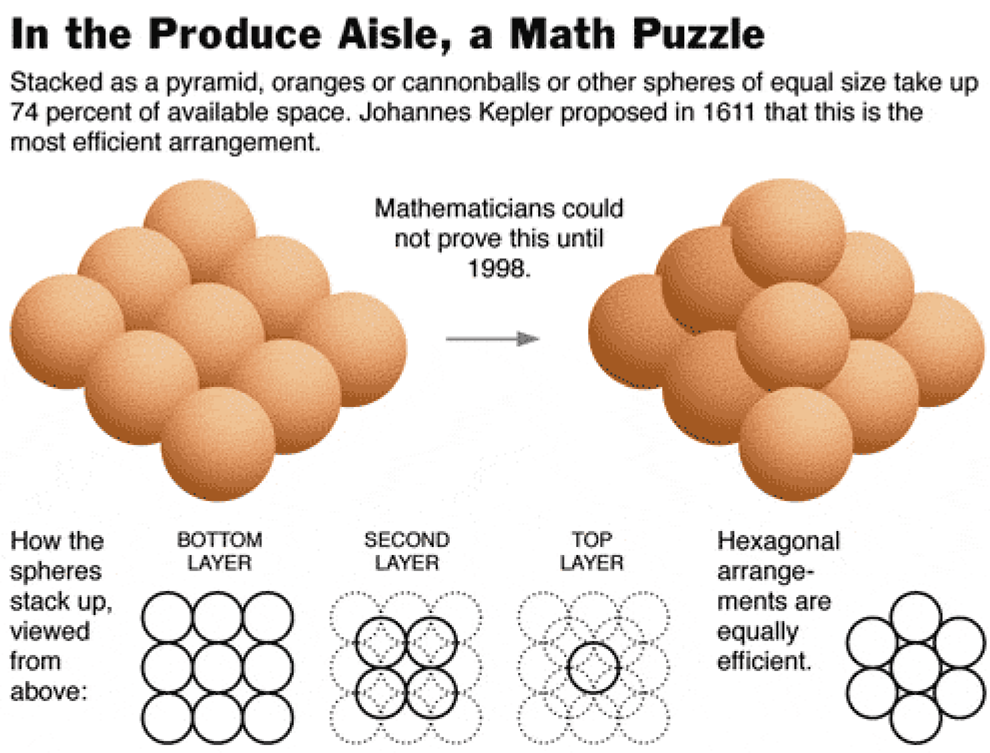


(***Source:*** **http://www.ugrad.math.ubc.ca/coursedoc/math100/notes/apps/maxmin.html**)

In his approach to determine the volume, Kepler approximated the barrel by a cylinder with base radius r of the base and height h; the conditions of the problem imply the equation $r^2 + h^2 = L^2$. He then looked for the values of r and h giving the largest volume V when L (which determines the cost) is held constant. Using differential calculus one can show that the relation between L and h must be $3L^2 = h^2$, which is the answer that Kepler found using less refined methods. He also observed that the shapes of the wine barrels were close to this optimal value — in fact, so close that he could not believe this was a coincidence. Of course, the manufacturing processes then were less uniform than they are now, so is was unlikely that all barrels satisfied this mathematical specification precisely, but Kepler further noted that if a barrel deviated slightly from the optimal ratio this would have little effect on the volume because the volume function changes very slowly near its maximum.

Other contributions of Kepler

We have already mentioned Kepler's work on semi – regular polyhedra in Unit 3; specifically, this appears at the end of **(3.D)**. He is also known for his so – called ***sphere packing problem***, which was first stated in 1611 and conjectures that the most efficient way of packing solid disks into a box is the usual method in which oranges (or other solid round objects) are stacked on top of each other (see the figure below).



(Source: http://www.math.binghamton.edu/zaslav/Nytimes/+Science/+Math/sphere-packing.20040406.gif)

Although this conjecture intuitively seems very likely to be true and strong partial results were obtained over the years, a complete proof has been extremely elusive. As noted on pages 750 – 751 of Burton, T. Hales (1958 –) announced a proof of this result in

1998. Many prominent experts in the area are confident that Hales has given a valid proof, but the argument depends on massive amounts of computer calculation, and current estimates are that it will take another decade or so before the accuracy of the computer calculations can be suitably verified. Some online references for further information are given below; the article from the ***New York Times*** is particularly informative (newspaper articles on advanced mathematical topics are challenging to write and are often not highly reliable sources, but this is an exception).

http://www.maa.org/devlin/devlin_9_98.html

http://www.sciencenews.org/sn_arc98/8_15_98/fob7.htm

http://mathworld.wolfram.com/KeplerConjecture.html

http://www.math.binghamton.edu/zaslav/Nytimes/+Science/+Math/sphere-packing.20040406.html

The emergence of calculus

Since many of the basic facts in calculus were known before the work of Newton and Leibniz, it is natural to ask why they are given credit for inventing the subject. Others came very close to doing so; in particular, Barrow understood that the process for finding tangents (differentiation in modern terminology) was inverse to the process for finding areas (integration in modern terminology). In these notes we shall focus on the decisive advances that make the work of Newton and Leibniz stand out from the important, high quality results due to many of their excellent contemporaries.

1. Before Newton and Leibniz techniques for differentiating and integrating specific examples had been developed, but they were the first to set general notation and define general "algorithmic processes" for each construction. Earlier workers were not able to derive useful and general problem – solving methods.

2. Newton and Leibniz recognized the usefulness of differentiation and integration as general processes, not just as ***ad hoc*** methods to solve measurement and tangent problems in important special cases. Apparently no one (at least in Europe) had previously recognized the usefulness of calculus as a general mathematical tool.

3. With the exception of Barrow, the inverse relationship between differentiation and integration had not been clearly recognized in earlier work, and Newton and Leibniz were the first to formulate it explicitly and establish it in a logically convincing manner.

4. Both stated the main ideas and results of calculus algebraically, so that the subject was no longer an offshoot of classical Greek geometry but significantly broader in scope and poised to make fundamental contributions to many areas of knowledge.

5. Both were able to go much further than the Kerala school in India and K. Seki in 17th century Japan (both were discussed in Unit 6); neither of the latter had the advantage of newly formulated problems from recent and extensive scientific investigations, and the questions arising from these studies were crucial to determining the breadth and generality in the work of Newton and Leibniz.

Sections 8.3 and 8.4 of Burton contain a great deal of detail about the scientific and philosophical contributions of Newton (see pages 386 – 408) and Leibniz (see pages 410 – 430). In particular, the bitter and heated dispute about credit for discovering calculus is described there. We shall only summarize the points that are now generally accepted: The discoveries of Newton and Leibniz were essentially independent, and although Newton was the first to develop the subject, Leibniz published his version first. We should add that the discoveries by Newton and Leibniz took place around 1665 and 1673 respectively. Leibniz' work was published in 1684 while Newton's was published 1736, nearly a decade after his death. Rather than dwell on the dispute over priorities, we shall discuss a few substantive similarities and differences between the work of Newton and Leibniz.

Many of the similarities were already mentioned in the reasons why Newton and Leibniz are given credit for creating calculus. One additional similarity is that each used both differentiation and integration to solve difficult and previously unsolved problems. Both also proved many of the same basic results; ***e.g.***, the standard rules for differentiating functions, the Fundamental Theorem of Calculus, and the basic formal integration techniques which appear in calculus textbooks. On the other hand, Newton and Leibniz clearly had different priorities and these can be seen in some differences between their approaches and conclusions.

1. The standard binomial series expansion for $(1+x)^a$, where a is an arbitrary real number and $|x| < 1$, is solely due to Newton.
2. Newton used the words *fluxion* and *fluent* to denote the derivative and integral, and he denoted derivatives by placing dots over variables. Leibniz ultimately adopted the dx notation and the integral sign that have been standard for centuries.
3. Newton was primarily interested in the uses of calculus to study problems involving motion, while Leibniz' work and interests involved finding extrema and solving differential equations.
4. Newton discovered the rules and processes of calculus by a study of velocity and distance, while Leibniz did so via algebraic sums and differences.
5. Newton used infinitesimals as a computational means, while Leibniz used them directly.
6. Newton's priority was differentiation while Leibniz' was integration.
7. Newton stressed the use of infinite series to express functions, while Leibniz preferred solutions that could be written in finite terms.
8. Leibniz gave more general rules and more convenient notation.

Despite the clear differences in the Newton and Leibniz approaches to calculus, ***each perspective has advantages in certain situations***. For example, some functions cannot be expressed in finite terms involving standard functions from first year calculus but can be studied very effectively using infinite series as in Newton's treatment, while others cannot be studied using infinite series and thus are more compatible with Leibniz' viewpoint. Examples of each type appear in **http://math.ucr.edu/~res/history14c.pdf** and **http://math.ucr.edu/~res/history14d.pdf**.

The other important writings of Newton and Leibniz reflect some of the differences mentioned above. Leibniz wrote many lengthy and highly influential works on philosophy (which as usual are beyond the scope of this course), while Newton wrote several important books on the sciences. The latter include his best known work, *Philosophiæ naturalis principia mathematica (The Mathematical Principles of Natural Philosophy)*, which remains of the most important books in the sciences ever written. In this book he developed the laws of motion using calculus and used them to derive Kepler's laws of planetary motion. An extremely brief but informative summary of *Principia* is available at the following online site:

http://www.answers.com/topic/newton-s-principia

We have already mentioned some common aspects of Newton's and Leibniz' legacies with respect to calculus, and we shall conclude this discussion by mentioning some noteworthy differences:

1. Newton's applications of calculus ultimately determined the direction of subsequent work in mathematics and physics.
2. Leibniz' formulation of calculus ultimately determined how the mathematical aspects of this work were formulated (however, Newton's dot notation $\dot{x}$ for derivatives is still used sometimes in physics to denote derivatives with respect to time).

In other words, Newton's legacy is more about the sorts of scientific problems that calculus has considered during the past three to four centuries, while Leibniz' legacy is more about the way such problems are studied.

Other contributions of Newton and Leibniz

Both Newton and Leibniz made other substantial contributions to mathematics aside from their momentous work on calculus. We shall mention a few of these contributions here.

<u>Newton</u> — Codiscovered (with J. Raphson, 1648 – 1715) the ***Newton – Raphson method*** for numerically approximating solutions to complicated nonlinear equations, including irrational roots of higher degree polynomials. This is one of the best known methods for finding such solutions; there are descriptions of this method in many (most?) standard calculus textbooks. Examples of the method and a brief description are given on page 434 of Burton.

<u>Newton</u> — In an appendix to his major scientific work titled ***Opticks*** (published in 1704), he classified all curves in the coordinate plane which are defined by third degree polynomial equations in the coordinates x and y. This has been characterized as the first comprehensive study of curves and their properties since the work of Apollonius. Newton also did several other important pieces of algebraic work, including his generalization of the binomial formula (which we have already mentioned) and his codiscovery (with A. Girard) of certain important identities involving symmetric polynomials in n algebraically independent variables (a polynomial $p(x_1, \dots, x_n)$ is said to be ***symmetric*** if any permutation of the variables yields the same polynomial; for example, $x_1 + \dots + x_n$ and $x_1 \dots x_n$ are symmetric but x_1 and $x_1x_3 + x_2x_3$ are not). Here are some references for further information:

http://en.wikipedia.org/wiki/Cubic_plane_curve

http://mathworld.wolfram.com/CubicCurve.html

http://www.2dcurves.com/cubic/cubic.html

http://en.wikipedia.org/wiki/Newton%27s_identities

http://staff.jccc.net/swilson/planecurves/cubics.htm

Newton — Contributions to the calculus of finite differences, whose uses include describing recursively defined sequences like the Fibonacci numbers. Further information on finite differences and Newton's results can be found at the following online site:

http://en.wikipedia.org/wiki/Finite_differences

Leibniz — Early work on studying logic using algebraic methods to manipulate well – formed grammatical statements. This has been described as one of the most important advances in the theory of logic between the time of Aristotle and the more systematic work on symbolic logic in the 19th century by G. Boole (1815 – 1864) and others. Further information on the development of algebraic methods in logic can be found on pages 7 – 8 of the following:

http://math.ucr.edu/~res/math144/setsnotes1.pdf

On page 8 of the cited document there is a reference to pages 643 – 647 in the 6th Edition of Burton; the corresponding pages in the 7th Edition are 646 – 650.

Leibniz — A comprehensive formulation of the ***binary*** or base 2 numeration system. This expanded upon many earlier ideas about representing numbers or words by combinations of two symbols, including numeration systems in a few cultures and the ancient writings of Pingala from 2000 years earlier, and in fact Leibniz explicitly cited symbolism in the ancient Chinese ***I Ching*** (in modern Pinyin transliteration, **Yì Jīng**) as one forerunner of his work. The importance of binary numeration for electronic computing was recognized beginning in the nineteen thirties, most notably by C. Shannon (1916 – 2001). Here are some online references for further information:

http://en.wikipedia.org/wiki/I_Ching

http://en.wikipedia.org/wiki/Binary_numeral_system

http://www.math.grin.edu/~rebelsky/Courses/152/97F/Readings/student-binary

http://www.mathsisfun.com/binary-number-system.html

http://www.thocp.net/biographies/shannon_claude.htm

http://en.wikipedia.org/wiki/Claude_Shannon

Leibniz — Solutions of systems of linear equations, including the definition of the determinant of a square matrix and the standard formula for it as a polynomial function of its entries (K. Seki did similar things independently about one decade earlier).