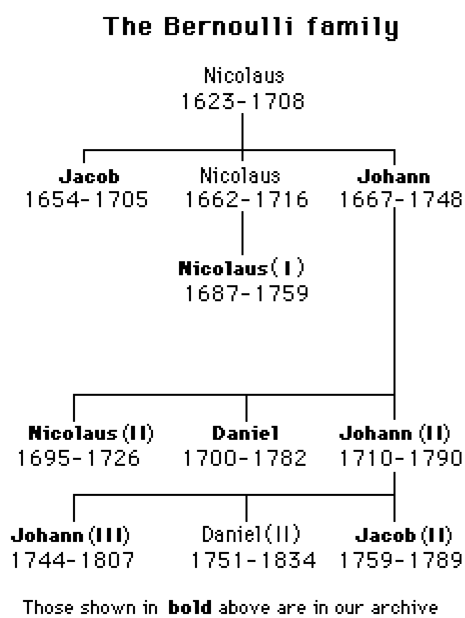
14. Calculus after Newton and Leibniz

(Burton, **9.3, 10.2, 11.3**)

We shall concentrate on points related to the material covered in first year calculus courses**.** Mathematics has continued to grow rapidly during the 300 + years following the invention of differential and integral calculus by Newton and Leibniz**,** but most of this history is well beyond the scope of the present course**.**

The Bernoulli family.



http://202.38.126.65/navigate/math/history/Diagrams/Bernoulli_family.gif

The first Bernoulli brothers — James or Jacob or Jakob or Jacques (1655 – 1705) and John or Johann or Jean (1667 – 1748) — made numerous important contributions to the subject soon after its development and publication**.** One small item worth noting is that the result known as L'Hospital's Rule was originally due to John Bernoulli but was sold to G. de L'Hospital (1661 – 1704) for an influential textbook the latter published in 1696**.** The Bernoullis were particularly effective at applying the methods of differential and integral calculus to analyze new types of mathematical questions that had previously been out of reach**.** One example is the ***brachistochrone problem*****,** which asks for the curve of quickest descent connecting two given points in a vertical plane**;** it turns out that a portion of the cycloid curve is a solution to this question**.** The Bernoullis made several other early contributions to the study of differential equations and mathematical probability, and they also gave some extremely important applications to physics**.** The areas they studied include optics, astronomy, fluid mechanics, wave motion, heat conduction and elasticity.

Solid analytic geometry. When plane analytic geometry was developed during the 17th century, researchers like Fermat, Schooten and P. de la Hire (1640 – 1718) were convinced that one could handle questions in $3-$ dimensional geometry similarly by adding one more coordinate and making suitable adjustments to various formulas, but the details of this program were not completed until the 18th century. Names associated with this work include J. Hermann (1678 – 1733), A. – C. Clairaut (1713 – 1765) and L. Euler.

Infinite series. We note first that the standard infinite power series for functions are named after B. Taylor (1685 – 1731) and C. Maclaurin (1698 – 1746); the usual attributions are an accident of history through no fault of Taylor or Maclaurin.

First year calculus books always mention that the infinite series

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots$$

converges, but often the value of the sum is not mentioned. In fact, Euler proved the unexpected relationship

$$\frac{\pi^2}{6}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\dots$$

using elaborate manipulations of infinite series. This was just one in a sequence of increasingly bold and dramatic summation formulas that Euler derived.

Such conclusions ultimately led Euler to carry out many speculative operations on infinite series that do **not** converge. In particular, he suggested that ½ is a reasonable value to take as the sum of the following divergent series:

$$1 \;+\; (-1) \;+\; 1 \;+\; (-1) \;+\; 1 \;+\; (-1) \;+\; 1 \;+\; (-1) \;+\; \dots$$

This may seem absurd, but it turns out that one can find some good mathematical justifications for attaching this value to the divergent series. Examples of this sort are somewhat artificial, but one can also construct important classes of physical problems involving infinite series with closely related convergence difficulties. Predictably, one eventually encounters supposed formulas that lead to contradictory answers, but in many cases it turns out that there are fairly large classes of infinite series which are divergent in the usual sense but can be viewed as "weakly convergent" in some precisely generalized sense. One particularly important example is summability in the sense of E. Cesàro (1859 – 1906), and in fact Cesàro's summation method yields Euler's "reasonable value" of ½ for the sum displayed above. Euler has been criticized for mindless formal manipulation in connection with his attempts to sum divergent series, but it is important to remember that he explicitly recognized the highly speculative nature of his thoughts on this issue, and his main objective was to see how far one could push the methods that proved to be so useful and reliable in other contexts (in other words, he was testing the robustness of the methods). **REFERENCE**

The preceding discussion leads naturally to the next topic:

The logical soundness of calculus

Aside from the questions on infinite series that we have just raised, there are even more important issues regarding calculus that were problematic during the 17th and 18th century. The most important of these was the use of infinitesimals. Not surprisingly there were many questions about the logical consistency of using objects that were smaller than any finite positive quantity but still positive. Proponents of calculus attempted to explain this concept, but such explanations did not really make much sense to some mathematicians of that time, even though the computational methods of Newton and Leibniz were getting the right answers, regardless of whether the explanations were understandable. Probably the most famous critique of infinitesimals was ***The Analyst***, by Bishop G. Berkeley (BARK – lee, 1685 – 1753); mathematicians and others realized the validity of his logical objections (as usual, it is beyond the scope of this course to assess his philosophical conclusions). Progress in mathematics continued at a rapid pace, but Berkeley's criticisms reinforced earlier views of many that calculus needed a more secure logical foundation. With the development of calculus, mathematics had moved into new territory, not just abstracting familiar ideas but also contributing new concepts of its own. It was also rapidly accepting an ever expanding collection of ideas and methods that were increasingly removed from ordinary experience. In order to handle such new concepts it is necessary to maintain very strict logical standards in order to compensate for the increased remoteness from sensory experience. Of course, the same applies to the elaborate manipulations with infinite series that mathematicians had been carrying out.

The resolution of the problems with infinitesimals led mathematicians to base calculus on the concept of ***limit***. This need had already been tentatively anticipated by Wallis and Gregory; J. L. D'Alembert (1717 – 1783) proposed a definition of limits, but the wording needed to be made more precise. The decisive step in this direction was due to A. – L. Cauchy (1789 – 1857). In particular, his textbook of 1821 included the concept of limit (a concept which had not appeared explicitly in the work of Newton or Leibniz) in a form very close the one in use today, and his definition of derivative is precisely the one used today. Cauchy also stressed that the definite integral should be defined as the limit certain algebraic sums and is independent of the definition of the derivative. It is from Cauchy's view of the integral that broad modern generalizations of this concept have developed. The full modern definition of limit using positive numbers δ and ε is due to K. T. W. von Weierstrass (1815 – 1897).

Despite their doubtful logical status, users of mathematics continued to work with infinitesimals, probably motivated by their relative simplicity, the fact that they gave reliable answers, and an expectation that mathematicians could ultimately find a logical justification for whatever was being attempted. Furthermore, the use of infinitesimals in calculus and other textbooks continued well into the 20th century for several reasons. Ultimately mathematicians succeeded in giving a logically rigorous justification for the concept of infinitesimals; in particular, during the nineteen sixties Abraham Robinson (1918 – 1974) used extensive machinery from abstract mathematical logic to show that one can in fact construct a number system with infinitesimals that satisfy the usual rules of arithmetic. However, the crucial advantage of Robinson's modern concept of infinitesimal — its logical soundness — is offset by the fact that, unlike 17th century infinitesimals, it is neither simple nor intuitively easy to understand.

The definition of limit was one step in strengthening the mathematical foundations for calculus. Mathematicians also came to realize that functions could behave in bizarre manners that they had not previously imagined, and for several reasons it was absolutely necessary to take such examples into account. Stronger logical justifications were needed for many basic points in calculus; for example, the fact that continuous functions on closed intervals take maximum and minimum values, and the Intermediate Value Property for continuous functions on open or closed intervals. Further thought was needed to understand the problems that arose if one was too casual when working with infinite series. Such questions were not just academic, for in fact they arise quickly in connection with real world problems like studying wave motion or heat conduction. A great deal of work was done to justify the basic concepts and results of calculus during the 19th century. **REFS?** Ultimately mathematicians realized that a secure logical foundation for calculus required a logically rigorous description of the real number system, which in turn required a theory of infinite sets. Both of these were accomplished during the second half of the 19th century. The necessary machinery for setting up the real number systems was mainly due to R. Dedekind (1831 – 1816), and the single most important early figure in the creation of set theory was G. Cantor (1845 – 1918).

Additional information on the historical development of set theory is presented on pages 4 – 14 of the following online reference:

http://math.ucr.edu/~res/math144/setsnotes1.pdf

Some of this material may be covered in the course if time permits.