Math 153
Spring 2010
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EXERCISES RELATED TO `history06*.pdf`, $* =$ `X,Y`

As in the earlier exercises, “Burton” refers to the Seventh Edition of the course text by Burton (the page numbers for the Sixth Edition may be off slightly).

- Burton, p. 231: 17, 18, 21, 22
- Burton, p. 263: 7, 14, 15

Additional exercises

1. Find all integers n with the following properties:

n leaves a remainder of 6 when divided by 9.
n leaves a remainder of 3 when divided by 4.

2. Find all integers n with the following properties:

n leaves a remainder of 2 when divided by 9.
n leaves a remainder of 6 when divided by 11.

3. Find all integers n with the following properties:

n leaves a remainder of 0 when divided by 3.
n leaves a remainder of 8 when divided by 13.

4. Find all integers n with the following properties:

n leaves a remainder of 1 when divided by 3.
n leaves a remainder of 1 when divided by 4.
n leaves a remainder of 2 when divided by 5.

5. Find all integers n with the following properties:

n leaves a remainder of 1 when divided by 2.
n leaves a remainder of 2 when divided by 3.
n leaves a remainder of 4 when divided by 5.
n leaves a remainder of 1 when divided by 7.

6. One solution to the Pell's equation $a^2 = 1 + 3b^2$ is $(a, b) = (7, 4)$.

(a) Prove that if (a, b) solves this equation then so does $(ac + 3bd, ad + bc)$ does too.

(b) Find two more solutions of the given equation such that a and b are positive integers.

7. Using the usual infinite series for $\arctan x$, derive an infinite series for $\pi/6 = \arctan(1/\sqrt{3})$ and estimate the error in approximating π using the first eight nonzero terms of this infinite series (in the 14^{th} century Madhara approximated π using this method).

8. Prove the following results which are due to Abu Abdallah Yaish ibn Ibrahim **Al-Umawi** (1400–1489):

(a) If n leaves a remainder of 5 when divided by 10, then n^2 leaves a remainder of 25 when divided by 100.

(b) If n leaves a remainder of 4 or 6 when divided by 10, then n^2 has the form $100m + 10q + 6$, where q is odd.

(c) If n leaves a remainder other than 4 or 6 when divided by 10, then n^2 has the form $100m + 10q + p$, where q is even.

(d) If n is a perfect cube, then n^3 leaves a remainder of 0, 1 or 6 when divided by 7. [*Hint:* Write $n = 7a + b$ where a and b are integers and $0 \leq b < 7$.]

(e) If n is a perfect cube, then n^3 leaves a remainder of 0, 1 or 8 when divided by 9.

9. Prove the following result which is also due to Al-Umawi: If n is a perfect square whose decimal expansion ends in 1, then $m = 1000a + 100b + 10c + 1$ where either b and $\frac{1}{2}c$ are both even or both odd.

10. Prove the following identity which was stated in the writings of Bhaskara II; assume that $a > b > 0$.

$$\sqrt{a \pm b} \quad = \quad \sqrt{\frac{a + \sqrt{a^2 - b}}{2}} \pm \sqrt{\frac{a - \sqrt{a^2 - b}}{2}}$$

11. The standard long division procedure taught in schools computes the quotient digit by digit, just like many of the algorithms used by Chinese mathematicians during the ancient and classical periods. The exercise below gives a formal justification for this method:

Suppose we are given an integer written in the form $10a + b$, where a, b are integers such that $a > 0$ and $0 \leq b \leq 9$. Let $d > 1$ be an integer. Assume that we have already found nonnegative integers p and r such that $a = dp + r$ where $0 \leq r < d$, and write $10r + b = dq + s$ where d and s are nonnegative integers with $0 \leq s < d$, so that

$$10a \;+\; b \quad = \quad (10p \;+\; q)d \;+\; s\ .$$

Prove that $q \leq 9$, so that $10p + q$ effectively gives the decimal expansion for the quotient to one more term. [*Hint:* The crucial given information is that $0 \leq r < d$ and $0 \leq b \leq 9 < 10$. Show that $dq + s < 10(r + 1)$ and $r + 1 \leq d$. Why does this imply that $q + (s/d)$ is less than 10?]