

EXERCISES RELATED TO history09.pdf

As in the earlier exercises, “Burton” refers to the Seventh Edition of the course text by Burton (the page numbers for the Sixth Edition may be off slightly).

- Burton, p. 326: 1, 3, 7, 8, 10, 12, 13

Additional exercises

1. (From Cardano’s *Ars Magna*) (a) Suppose that the cubic equation $x^3 + px + q = 0$ has real coefficients and has all real roots. Prove that the weighted sum of the roots (with multiple roots counted an appropriate number of times) is equal to zero.

(b) Suppose that r and s are positive roots of the cubic equation $x^3 + d = cx$. Prove that $r + s$ is a root of the cubic equation $x^3 = cx + d$.

2. (Also from Cardano’s *Ars Magna*) Suppose that a is a real root of the cubic equation $x^3 = cx + d$, where c and d are again positive real numbers. Prove that the numbers

$$\frac{a}{2} \pm \sqrt{\frac{4c^2 - 3a^2}{4}}$$

are roots of the equation $x^3 + d = cx$. Apply this to solve $x^3 + 3 = 8x$.

3. Suppose we are given the cubic equation $x^3 = cx + d$ where c and d are positive. Using calculus, show that if

$$\left(\frac{c}{3}\right)^3 > \left(\frac{d}{2}\right)^2$$

(so that the terms inside the cube root signs of the Tartaglia-Cardano cubic formula are not real numbers), then the cubic equation has three real roots. [*Hint:* Find x_M such that $f(x) = x^3 - cx - d$ takes a relative maximum value. Show that $x_M < 0$ and $f(0) < 0$. Explain why the limit of $f(x)$ as $x \rightarrow \pm \infty$ is equal to $\pm \infty$, and apply the Intermediate Value Theorem to show that there is one root less than x_M , one root between x_M and 0, and one positive root.]

4. Prove that the equation $x^3 + cx = d$ (again $c, d > 0$) has one positive root and no negative roots.

5. In history02.pdf we mentioned that $\cos 20^\circ$ is a root of the cubic polynomial $8y^3 - 6y - 1 = 0$. Find a nontrivial cubic polynomial with integral coefficients which has $\cos 40^\circ$ as a root. [*Hint:* Let $y = \cos 20^\circ$ and express $\cos 40^\circ$ as a polynomial in y .]

6. Using the half angle formulas for sines and cosines, find a nontrivial quartic polynomial with integral coefficients which has $\cos 22\frac{1}{2}^\circ$ and $\sin 22\frac{1}{2}^\circ$ as two of its roots.

7. Find a nontrivial quartic polynomial with integral coefficients which has $\sqrt{3 - 2\sqrt{2}}$ as one of its roots.