Math 153
Spring 2010
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EXERCISES RELATED TO `history11.pdf`

As in the earlier exercises, "Burton" refers to the Seventh Edition of the course text by Burton (the page numbers for the Sixth Edition may be off slightly).

- Burton, p. 361: 7, 8

Additional exercises

1. Justify the assertions regarding Fermat's result on coordinate geometry which was cited in `history11.pdf`: *Given any finite number of fixed lines* $\{\mathcal{L}_1, \cdots\}$, *let* $d_i(\mathbf{p})$ *denote the distance from* $\mathbf{p}$ *to* $\mathcal{L}_i$ *(measured along the perpendicular from* $\mathbf{p}$ *to the line). Then the set of points* $\mathbf{p}$ *such that* $\sum d_i^2 = K$, *for some constant* K, *is an ellipse.*

The first step is to show that the distance from (x, y) to the line with equation $au + bv + c = 0$ is given by

$$\frac{|ax + by + c|}{\sqrt{a^2 + b^2}} .$$

Next, explain why the sum of the squares of the distances is a nontrivial quadratic polynomial in x and y, so that the set of points is a conic. Finally, show this conic is an ellipse; one easy way is to check that the equation defines a bounded subset of the coordinate plane; in other words, one can find a positive constant M such that for all (x, y) satisfying the equation we have $|x|, |y| \leq M$ (note that ellipses are bounded but hyperbolas and parabolas are not).

2. Fill in the details for the following solution to Pappus Line Problem (compare Burton, page 368) using coordinate geometry: Namely, if we are given four lines $\mathcal{L}_i$ and four angles α_i, let u_i denote the distance from a point $\mathbf{p}$ to $\mathcal{L}_i$ measured along a line $\mathcal{M}_i$ which makes an angle of α_i units (degrees, radians, etc.) with $\mathcal{L}_i$. Then the set of all points $\mathbf{p} = (x, y)$ such that $u_1 u_2 = k \cdot u_3 u_4$, for some positive constant K, is contained in a union of two conics, two lines, or a line and a conic.

(a) Using input from the previous exercise, explain why the set of all such points satisfies the following equation:

$$(a_1 x + b_1 y + c_1) \cdot (a_2 x + b_2 y + c_2) \quad = \quad \pm K \cdot (a_3 x + b_3 y + c_3) \cdot (a_4 x + b_4 y + c_4)$$

where the line $\mathcal{L}_i$ is defined by an equation of the form $a_i x + b_i y + c_i = 0$. Explain why there is a need for a sign (notice that absolute values arise in the distance formula).

(b) Explain why the coordinates of all points under consideration must satisfy one of two polynomial equations in two variables of degree at most two.

(c) (Optional) Explain why these polynomial equations involve nontrivial polynomials of two variables if the four lines are distinct and $K > 0$. [*Note:* We need to show that $(a_1 x + b_1 y + c_1) \cdot (a_2 x + b_2 y + c_2)$ and $(a_3 x + b_3 y + c_3) \cdot (a_4 x + b_4 y + c_4)$ are not scalar multiples of each other; one quick way of doing this is to use the unique factorization property for polynomials in x and y with real coefficients. Proofs for this result appear in many abstract algebra textbooks at the upper undergraduate or introductory graduate level.]

(d) Consider the four lines (in the given order) defined by the equations $x = 1$, $y = 2$, $x = 3$ and $y = 4$, and let $K = \pm 2$. Show that the equations $(x - 1)(y - 2) = \pm 2(x - 3)(y - 4)$ define different hyperbolas (this shows that sometimes one actually needs two distinct conics; this is not clear from the discussion in Burton).

3. Formulate and verify a similar result for three lines instead of four, in which the conclusion involves the equation $u_1 u_2 = k \cdot u_3^2$.