Math 153
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SOLUTIONS TO EXERCISES FROM `math153exercises01.pdf`

As usual, "Burton" refers to the Seventh Edition of the course text by Burton (the page numbers for the Sixth Edition may be off slightly).

Problems from Burton, p. 28

3. The fraction 1/6 is equal to 10/60 and therefore the sexagesimal expression is 0;10.

To find the expansion for 1/9 we need to solve $1/9 = x/60$. By elementary algebra this means $9x = 60$ or $x = 6\frac{2}{3}$. Thus

$$x \quad = \quad \frac{6}{60} + \frac{2}{3} \cdot \frac{1}{60} \quad = \quad \frac{6}{60} + \frac{40}{60} \cdot \frac{1}{60}$$

which yields the sexagsimal expression $0;6,40$ for 1/9.

Finding the expression for 1/5 just amounts to writing this as 12/60, so the form here is 0;12.

To find 1/24 we again write $1/24 = x/60$ and solve for x to get $x = 2\frac{1}{2}$. Now $\frac{1}{2} = \frac{30}{60}$ and therefore we can proceed as in the second example to conclude that the sexagesimal form for 1/24 is 0;2,30.

One proceeds similarly for 1/40, solving $1/40 = x/60$ to get $x = 1\frac{1}{2}$. Much as in the preceding discussion this yields the form 0;1,30.

Finally, the same method leads to the equation $5/12 = x/60$, which implies that 5/12 has the sexagesimal form 0;25.■

4. We shall only rewrite these in standard base 10 fractional notation. The answers are in the back of Burton.

(a) The sexagesimal number 1,23,45 is equal to $1 \times 3600 \ + \ 23 \times 60 \ + \ 45$.

(b) This number is equal to

$$12 \ + \ \frac{3}{60} \ + \ \frac{45}{60 \times 60} \ .$$

(c) This number is equal to the previous one divided by 60.

(d) This number is simply equal to the first one divided by 60.

5. The general rule is to shift the semicolon one place to the right, so in this particular example the product is 1,23,45;6.

Problems from Burton, p. 50

8. This is simply a matter of verifying an identity and checking it against the list in the book, The expression for $\frac{2}{7}$ follows from $7 + 1 = 8$, the expression for $\frac{2}{35}$ follows from $35 = 7 \times 5$ and $7 + 5 = 12$, and finally the expression for $\frac{2}{91}$ follows from $91 = 13 \times 7$ and $7 + 13 = 20$.■

13. This uses the identity

$$\frac{1}{m} \quad = \quad \frac{1}{m+1} \quad + \quad \frac{1}{m(m+1)}$$

which can be checked directly. If n divides $m+1$ this means that $m+1 = n \cdot q$ for some q, If we multiply both sides of the displayed equation by n, substitute the factorization of $m+1$ into the equation and simplify, we then obtain the identity

$$\frac{n}{m} \quad = \quad \frac{1}{q} \;+\; \frac{1}{q\,m} \;.\blacksquare$$

Problem from Burton, p. 62

9. (a) If we apply the formula in the exercise to each of the triangles $\triangle ADC$, $\triangle DCB$, $\triangle CBA$, $\triangle BAD$, we find that the sum of their areas is

$$\tfrac{1}{2}\,dc \sin D \;+\; \tfrac{1}{2}\,cb \sin C \;+\; \tfrac{1}{2}\,ba \sin B \;+\; \tfrac{1}{2}\,ad \sin A \;.$$

Let X be the point where AC and BD meet. Then we have the following equations:

$$\begin{aligned}
\mathbf{area}(\triangle ADC) \quad &= \quad \mathbf{area}(\triangle AXD) \;+\; \mathbf{area}(\triangle DXC) \\
\mathbf{area}(\triangle DCB) \quad &= \quad \mathbf{area}(\triangle DXC) \;+\; \mathbf{area}(\triangle CXB) \\
\mathbf{area}(\triangle CBA) \quad &= \quad \mathbf{area}(\triangle CXB) \;+\; \mathbf{area}(\triangle BXA) \\
\mathbf{area}(\triangle BAD) \quad &= \quad \mathbf{area}(\triangle BXA) \;+\; \mathbf{area}(\triangle AXD)
\end{aligned}$$

Now we also know that the area $\mathbf{A}$ of the quadrilateral $ABCD$ is equal to

$$\mathbf{area}(\triangle DXC) \;+\; \mathbf{area}(\triangle CXB) \;+\; \mathbf{area}(\triangle BXA) \;+\; \mathbf{area}(\triangle AXD)$$

and if we add the previous four lines we find that the sum of the areas of $\triangle ADC$, $\triangle DCB$, $\triangle CBA$ and $\triangle BAD$ is equal to $2\,\mathbf{A}$. Substituting this into the first expression in this exercise we obtain the formula

$$2\,\mathbf{A} \quad = \quad \tfrac{1}{2}\,dc \sin D \;+\; \tfrac{1}{2}\,cb \sin C \;+\; \tfrac{1}{2}\,ba \sin B \;+\; \tfrac{1}{2}\,ad \sin A$$

and if we divide both sides of this equation by 2 we obtain the area formula given in the exercise.$\blacksquare$

(b) Since the sine of an angle between $0°$ and $180°$ is ≤ 1, the formula in (a) implies that

$$\mathbf{A} \quad \leq \tfrac{1}{4}(dc \;+\; cb \;+\; ba \;+\; ad)$$

and the inequality in the exercise follows because the right hand side is equal to $\frac{1}{4}\,(a+c)\,(b+d)$. Furthermore if any of the vertex angles $\angle\,A$, $\angle\,B$, $\angle\,C$, $\angle\,D$ is NOT a right angle, then the sine of that angle is strictly less than one and this implies the inequality must be strict.$\blacksquare$

Problems from Burton, p. 71

4. We need to solve the equations $x + y = 10$ and $xy = 16$ for x and y. Given that this is a second degree system we can expect to find two solutions but for a meaningful solution of the original physical problem both x and y must be positive.

The hint suggests using the formula

$$(x - y)^2 \quad = \quad (x + y)^2 \; - \; 4\,xy$$

and if we substitute the given equations into the right hand side we obtain the equation

$$(x - y)^2 \quad = \quad 10^2 \; - \; 4\,16 \quad = \quad 36 \; .$$

Thus we have $x - y = \pm\,6$. If $x - y = +6$, the solution we obtain is $x = 8, y = 2$, while if $x - y = -6$ we obtain the solution $y = 8, x = 2$. In particular, this means that we have a rectangle that is 8 by 2.

6. If we allow solutions that are arbitrary complex numbers, then the usual methods of college algebra show that a system of two equations in x and y, with one linear and the other quadratic (like $xy =$ constant), has at most two solutions; this is true because the equations $ax + by = P$ and $xy = Q \neq 0$ can be rewritten in the form

$$ax \; + \; \frac{bQ}{x} \quad = \quad P \quad \text{or} \quad ax^2 \; - \; Px \; + \; bQ \quad = \quad 0 \; .$$

(a) In this case use the formula

$$(x - y)^2 \quad = \quad (x + y)^2 \; - \; 4\,xy$$

to find $(x + y)^2$. Specifically, we have $36 = (x + y)^2 - 64$, which leads to $x + y = \pm\,10$. If we are only looking for positive solutions to the system equations, then the last equation is solvable only if $x + y = 10$. Since $x - y = 6$, it follows that $x = 8$ and $y = 2$ is the unique positive solution.

On the other hand, as noted above there is a second solution; in general, for the given type of system one can read off the second solution from the first, for if $x = a,\; y = b$ solves the given system of equations then so does $x = -b,\; y = -a$ (check this!). Therefore there are two real solutions of the original system, and they are $(x, y) = (8, 2)$ and $(-2, -8)$.■

(b) Use the same formula as before, but substitute the numerical values for this specific problem to obtain the equation $16 = (x + y)^2 - 84$. Once again this leads to $x + y = \pm\,10$, and the positive solution is $(x, y) = (7, 3)$, with an additional real solution of $(x, y) = (-3, -7)$.■

(c) Once again use the formula, this time obtaining the equation $(x - y)^2 = 64 - 60 = 4$. Thus we have $x - y = \pm 2$, so that the positive solution for this problem is $(x, y) = (5, 3)$. In this case, the system of equations is symmetric in x and y, so that if $(x, y) = (a, b)$ solves the system then so does $(x, y) = (b, a)$, and hence the second solution is given by $(3, 5)$.■

13. (a) The hint on page 68 of the text seems wrong, and since one already knows x it is reasonable to approach this by substituting the first equation into the second. This yields the equation

$$30\,y \; - \; (30 - y)^2 \quad = \quad 500$$

which after expansion and simplification reduces to

$$y^2 \; - \; 90\,y \; + \; 1400 \quad = \quad 0\ .$$

The roots of this equation are $y = 70$ and $y = 20$, and as noted before we are given $x = 30$.■

(b) Here we follow the hint on page 68 of the text and subtract the square of the first equation from twice the second. The square of the first equation has the form $x^2 + 2\,xy + y^2 = 2500$. If we subtract this from $2\,x^2 + 2\,y^2 + 2\,(x - y)^2 = 2800$ we obtain the following:

$$3\,(x-y)^2 \quad = \quad x^2 \; - \; 2\,xy \; + \; y^2 \; + \; (x-y)^2 \quad = \quad 300$$

This implies $x - y = \pm\,10$. Combining these with the original equation $x + y = 50$, we obtain the solutions $(x, y) = (30, 20)$ and $(20, 30)$ depending upon the sign $\pm$.■

(c) In this problem we also follow the hint and substitute $(x + y)^2 = (x - y)^2 + 4\,xy$ and $xy = 600$ into the equation $(x + y)^2 + 60\,(x - y) = 3100$. This yields the following quadratic equation in $(x - y)$:

$$(x-y)^2 \; + \; 60\,(x-y) \; - \; 700 \quad = \quad 0$$

The roots of this equation are $x - y = 10$ and -70. If we substitute this into $xy = 600$ and solve we obtain the solutions $(x, y) = (30, 20)$ and $(-20, -30)$ when $x - y = 10$, and when $x - y = -70$ we obtain the solutions $(x, y) = (-35 - 5\sqrt{73},\, 35 - 5\sqrt{73})$ and $(-35 + 5\sqrt{73},\, 35 + 5\sqrt{73})$.■

SOLUTIONS TO ADDITIONAL EXERCISES

0. We shall first give a brief explanation of how one can work this sort of problem. The idean is to take the base 10 numbers and write them as

$$W \times 16^3 \; + \; X \times 16^2 \; + \; Y \times 16 \; + \; Z$$

where $W,\ X,\ Y,\ Z$ are integers between 0 and 15. Given a number n in the range (so $n < 256^2 = 65,536$) the first step is to do long division and express $n = 16p + Z$. Next, we write $p = 16q + Y$ and $q = 16r + X$. By construction it will follow that $q < 256$, so that r is an integer between 0 and 15, and therefore $r = W$.

If we carry out the arithmetic in this case for $n = 64206$, when we divide by 16 we get an integral quotient of $p = 4012$ with a remainder of $Z = 14$. At the next step, when we divide $p = 4012$ by 16, we get an integral quotient of $q = 250$ with a remainder of $Y = 12$. Proceeding to the third step, when we divide $q = 250$ by 16, we get an integral quotient of $r = 15$ with a remainder of $X = 12$. But now we know that $W = r = 15$. Thus we have

$$(W, X, Y, Z) \quad = \quad (15, 10, 12, 14)$$

and if we convert this sequence into hexadecimal notation using A = 10, *etc.* the sequence can be written in hexadecimal form as (F, A, C, E), so for this example the hidden word is FACE.

Using this method, one can show that the hexadecimal equivalents of 57069, 51966, 61453, 3499, 3071, and 4013 are respectively given by DEED, CAFE, F00D, DAB, BFF, and FAD.■

1. For the first example, we have $165 \; = \; 1 \times 12^2 + 1 \times 12 + 9$, so the dozenal expansion is 119.

For the second example, we have $343 = 2 \times 12^2 + 4 \times 12 + 7$, so the dozenal expansion is 247.

For the third example, we have $666 = 4 \times 12^2 + 7 \times 12 + 6$, so the dozenal expansion is 426.

For the fourth example, we have $998 = 6 \times 12^2 + 11 \times 12 + 2$, so the dozenal expansion is 6#2.

In an additional example, we have $265 = 1 \times 12^2 + 10 \times 12 + 1$, so the dozenal expansion is $1 * 1$.■

2. We shall do these in order.

To find the sexagesimal form for 2/9 we have to write it as $x/60$. We can find x by the usual method for solving proportion equations: $9x = 2 \times 60 = 120 \Longrightarrow x = 120/9$. which translates to $x - 13\frac{1}{3}$. Now $1/3 = 20/60$, so this means we have

$$\frac{2}{9} = \frac{13}{60} + \frac{20}{60 \times 60} = 0; 13, 20 \ .$$

We approach 1/25 in the same way. The solution to the equation $1/25 = x/60$ is $x = 2\frac{2}{5}$. Since $\frac{2}{5} = 24/60$ we have

$$\frac{1}{25} = \frac{2}{60} + \frac{24}{60 \times 60} = 0; 2, 24 \ .$$

For the next two we have fractions that are clearly less than 1/60, so we should start with $60 \times 60 = 3600$ instead. Thus we want to start by solving $1/100 = x/3600$. This has the solution $x = 36$ and therefore the sexagesimal form is $0; 0, 36$.

Finally, for the last one we begin by solving $1/125 = x/3600$, and we find in this case that $x = 28\frac{4}{5}$. Now $4/5 = 48/60$ so we must have

$$\frac{1}{125} = \frac{28}{60 \times 60} + \frac{48}{60 \times 60 \times 60} = 0; 0; 28, 48 \ .■$$

3. By definition this is equal to

$$1 + \frac{24}{60} + \frac{51}{60 \times 60} + \frac{10}{60 \times 60 \times 60} = 1 + \frac{24 \times 3600 + 51 \times 60 + 10}{60 \times 60 \times 60} =$$

$$1 + \frac{86400 + 3060 + 10}{60 \times 60 \times 60} = 1 + \frac{89470}{216000} \ .$$

If we compute this number in decimals, we see that it is equal to 1.41421296296296296... and if we compare this to $\sqrt{2} = 1.4142356...$ we see that this must have been an approximation to the square root of 2 that is accurate to four (of our) decimal places.■

4. Let's start with $L = 2$. How many ways are there of writing a fraction r satisfying $0 < r < 1$ as a sum of two unit fractions? If we have an equation of the form

$$r = \frac{1}{a} + \frac{1}{b}$$

where for the sake of definitess we shall take $a < b$, then we have inequalities

$$r > \frac{1}{a} > \frac{r}{2}$$

which imply that

$$\frac{1}{r} \;<\; a \;<\; \frac{2}{r}\,.$$

Now there are only finitely many choices of integers a for which this inequality is true. For each such a consider the remainder $r - \frac{1}{a}$. This also lies between 0 and 1. If it is a unit fraction of the form $1/b$, then we have an Egyptian fraction expansion

$$r \;=\; \frac{1}{a} + \frac{1}{b}$$

but then again the remainder need not have this form. Regardless of whether or not it does, for each choice of a there is at most one way of writing r in Egyptian form such that $1/a$ is one of the terms. This means that there can only be finitely many Egyptian fraction expansions whose length L is equal to 2.

To prove the result for all L using finite induction, we need to show that if the conclusion is true for expansions of length L then it is true for expansions of length $L+1$. So suppose we have a value of L for which the conclusion is known to be true. If we are given an Egyptian expansion of r with length $L+1$, one of the terms in this expansion, say $1/a$, is larger than all the others. As in the case $L = 2$, this means that

$$\frac{1}{a} \;>\; \frac{r}{L+1}$$

for otherwise every one of the summands would be less than or equal to the right hand side, and only one of the $L+1$ terms could be equal to this. In such a situation the entire sum must be strictly less than r. Armed with the above inequality for the largest term in the expansion, we proceed as follows: Combining the displayed inequality with the basic relation $\frac{1}{a} < r$, we conclude that

$$\frac{1}{r} \;<\; a \;<\; \frac{L+1}{r}$$

and see that there are only finitely many possibilities for the largest term in an Egyptian expansion of length $L+1$. As before, for each such a consider the remainder $r - \frac{1}{a}$. This also lies between 0 and 1. By the hypothesis on expansions of length L, for each choice of a there are only finitely many ways of expanding the remainder as an Egyptian fraction of length L.

Suppose now that we fix a and consider the finite collection of expansions

$$r \;=\; \frac{1}{a} + \sum_{j=1}^{L} \frac{1}{n_j}$$

given by the previous paragraph. Every Egyptian fraction expansion of r containing the term $1/a$ is in this list; in fact, there may be other non-Egyptian type expansions in the list because it is possible that there is a summand $\frac{1}{a}$ in the sum of L terms, but in any case we see that there are only finitely many ways of writing r as an Egyptian fraction of length $L+1$ such that $1/a$ is one of the terms. But there are only finitely many options for a, so this means there can only be finitely many ways of expressing r by an Egyptian fraction expansion of length $L+1$. This proves the inductive step, and therefore the conclusion is true for all $L \geq 2$.■

5. We shall try to do the first few these using the Greedy Algorithm.

The largest unit fraction less than $\frac{2}{11}$ can be found by looking for the first integer which is greater than the reciprocal $\frac{11}{2} = 5\frac{1}{2}$. This integer is 6. Therefore the Greedy Algorithm gives $\frac{1}{6}$

as the first term and proceeds to consider the remainder. But $\frac{2}{11} - \frac{1}{6} = \frac{1}{66}$ so the we obtain the expansion $\frac{2}{11} = \frac{1}{6} + \frac{1}{66}$ right away.

Next consider $\frac{3}{11}$. In this case the Greedy Algorithm gives $\frac{1}{4}$ as the first term, and we compute the remainder to be $\frac{3}{11} - \frac{1}{4} = \frac{1}{44}$, so that $\frac{3}{11} = \frac{1}{4} + \frac{1}{44}$ in this case.

Now consider $\frac{4}{11}$, in which case the Greedy Algorithm yields $\frac{1}{3}$ as the first term and the remainder is $\frac{1}{33}$. Thus we have $\frac{4}{11} = \frac{1}{3} + \frac{1}{33}$ in this case.

In the case of $\frac{5}{11}$, the Greedy Algorithm still yields $\frac{1}{3}$ as the first term and the remainder is $\frac{4}{33}$. The latter is equal to $\frac{1}{11} + \frac{1}{33}$, and thus we have $\frac{5}{11} = \frac{1}{3} + \frac{1}{11} + \frac{1}{33}$ in this case.

For $\frac{3}{11}$, the Greedy Algorithm yields $\frac{1}{2}$ as the first term and the remainder is $\frac{1}{22}$. Thus we have $\frac{6}{11} = \frac{1}{2} + \frac{1}{22}$ in this case.

Turning to $\frac{7}{11}$ a first application of the Greedy Algorithm yields $\frac{7}{11} = \frac{1}{2} + \frac{3}{22}$. Rather than proceed to apply the Greedy Algorithm directly to the remainder of $\frac{3}{22}$, let's take the expansion we had for $\frac{3}{11}$ and multiply it by $\frac{1}{2}$ to obtain $\frac{3}{22} = \frac{1}{8} + \frac{1}{88}$. We then get the expansion $\frac{7}{11} = \frac{1}{2} + \frac{1}{8} + \frac{1}{88}$ in this case.

We can dispose of the remaining cases similarly. For $\frac{8}{11}$, combine $\frac{8}{11} = \frac{1}{2} + \frac{5}{22}$ and $\frac{5}{22} = \frac{1}{6} + \frac{1}{22} + \frac{1}{66}$ to obtain $\frac{8}{11} = \frac{1}{2} + \frac{1}{6} + \frac{1}{22} + \frac{1}{66}$ in this case.

Similarly, for $\frac{9}{11}$, combine $\frac{9}{11} = \frac{1}{2} + \frac{7}{22}$ and $\frac{7}{22} = \frac{1}{4} + \frac{1}{16} + \frac{1}{176}$ to obtain $\frac{9}{11} = \frac{1}{2} + \frac{1}{4} + \frac{1}{16} + \frac{1}{176}$.

Finally, for $\frac{10}{11}$, combine $\frac{10}{11} = \frac{1}{2} + \frac{9}{22}$ and $\frac{9}{22} = \frac{1}{4} + \frac{1}{8} + \frac{1}{32} + \frac{1}{352}$ to obtain $\frac{10}{11} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{32} + \frac{1}{352}$.■

6. (a) If $0 < u < v$, then $0 < u^2 < v^2$ and $0 < u^3 < v^3$. Adding these inequalities, we see that $0 < u^3 + u^2 < v^3 + v^2$. — Note that if we considered the function $y^3 - y^2$ instead, then one can use calculus to show that if $c > 0$ then the equation $y^3 - y^2 = c$ has a unique positive solution such that $y > 1$ (one can use calculus to show that the function has a positive derivative and is strictly increasing for $y > \frac{2}{3}$, and the value of the function is positive for $y > 1$).

(b) Using the change of variables we may write $k^3y^3 + bk^2y^2 = c$, which is equivalent to

$$y^3 \ + \ \frac{b}{k}\, y^2 \ = \ \frac{c}{k^3}$$

and we can make the coefficient of y^2 equal to 1 if we take $k = b$. So the change of variables is simply $x = by$.

(c) The change of variables is $x = y - a$, and thus the equation becomes

$$(y - a)^3 \ + \ b(y - a)^2 \ + \ c(y - a) \ + \ d \ = \ 0\ .$$

If we expand everything in sight we obtain a monic third degree polynomial in y, and the coefficient of y in this expression is given by $3a^2 - 2ab + c$, so the first degree term will vanish if a is a root of $3a^2 - 2ab + c = 0$. By the Quadratic Formula we have the following formula for a:

$$\frac{2b \pm \sqrt{4b^2 - 12c}}{6} \ = \ \frac{b \pm \sqrt{b^2 - 3c}}{3}$$

Note that these values of a are not necessarily real.■

7. (i) Since supplementary angles have the same sines and all vertex angles are either supplementary to $\angle A$ or have the same measure as $\angle A$, it follows that $\sin A = \sin B = \sin C = \sin D$. Thus the formula reduces to

$$\mathbf{A} \ = \ \tfrac{1}{4}\,(a + c)\,(b + d)\,\sin\theta\ .$$

(ii) The formula says that the ratio of the actual area to the formula area is equal to $\sin\theta$. For a 60° angle this sine is equal to $\frac{1}{2}\sqrt{3}$, and therefore the ratio of the actual to formula area in this case is also equal to $\frac{1}{2}\sqrt{3}$. This is approximately 87 per cent of the value predicted by the incorrect formula.■