Math 153
Spring 2010
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## SOLUTIONS TO EXERCISES FROM `math153exercises05.pdf`

As usual, "Burton" refers to the Seventh Edition of the course text by Burton (the page numbers for the Sixth Edition may be off slightly).

### Problems from Burton, p. 231

**4.** More generally, consider the system of equations $x^2 + y^2 = B$ and $x + y = A$ where $A$ and $B$ are (usually positive) integers. Since $x^2 + 2xy + y^2 = (x + y)^2 = A^2$, we can rewrite this system as $x + y = A$ and $2xy = A^2 - B$. If we are looking for positive solutions, then we might as well restrict attention to cases where $B < A^2$.

If we write $C = \frac{1}{2}(A^2 - B)$, then the system of equations implies that $y = C/x$ and

$$x + \frac{C}{x} = A$$

so that $x$ must satisfy the quadratic equation $x^2 - Ax - C = 0$. The Quadratic Formula implies that there is a positive rational solution to this equation if and only if $A^2 + 4C = 3A^2 - 2B$ is a perfect square. This is the case if $A = 20$ and $B = 208$ as in Burton, for we then have $3A^2 - 2B = 784 = 28^2$; more generally, one also obtains solutions if $A = 20k$ and $B = 208k^2$ for some positive integer $k$ (then the difference is $(29k)^2$). For the choices of $A = 20$ and $B = 208$ in Burton, we can now solve for $x$ and $y$ to conclude that $x = 12$ and $y = 8$.

In fact, there are many different ways of choosing integers $A$ and $B$ so that $3A^2 - 2B$ is a perfect square. Given an arbitrary positive integer $A$ and some other positive integer $D > A$ such that $(i)$ $A + D$ is even, $(ii)$ $D^2 < 3A^2$, the number $3A^2 - D^2$ will always be an even integer (check this!), and we can take $B$ to be half of this difference. To show this is consistent with our previous condition on $B$, we need to check that $3A^2 - D^2 = 2B < 2A^2$, but this follows because $D > A$, so that $A^2 < D^2$ and $3A^2 - D^2 < 3A^2 - A^2 = 2A^2$.

**13.** By the theorem on pages 215–216 of Burton, the equation $ax + by = c$ with integral coefficients has integral solutions if and only if the greatest common divisor $d$ for $a$ also divides $c$.

$(a)$ The greatest common divisor of $6 = 2 \times 3$ and $51 = 17 \times 3$ is 3, but 22 is not divisible by 3, so there is no integral solution for the equation.■

$(b)$ The greatest common divisor of $14 = 7 \times 2$ and $33 = 3 \times 11$ is 1, so the equation does have integral solutions.■

$(c)$ The greatest common divisor of $14 = 7 \times 2$ and $35 = 7 \times 5$ is 7, and $91 = 13 \times 7$, so the equation does have integral solutions.■

**14.** $(b)$ We shall first consider rational solutions to the equation $24x + 138y = 138$ and then try to determine when these yield integral solutions; we also want to describe all such solutions in a reasonable fashion.

The greatest common divisor of $24 = 8 \cdot 3$ and $138 = 2 \cdot 3 \cdot 23$ is 6, so in fact we can use the Euclidean Algorithm to find some pair of integers $a, b$ such that $24a + 138b = 6$; an explicit choice is given by $(a, b) = (6, -1)$. Since $18 = 6 \cdot e$, we have the solution $(x_0, y_0) = (18, -3)$ to the original equation. We now need to see what can be said about the general solution. By subtraction of one equation for another, we see that if $(x, y)$ is an arbitrary solution then

$$24(x - x_0) \ + \ 138(y - y_0) \quad = \quad 0$$

which reduces to $4(x - x_0) + 23(y - y_0) \ = \ 0$. The integral solutions to this equation are given by all multiples of $(23, -4)$, and therefore the general solution is given by $x = x_0 + 23k = 18 + 23k$ and $y = y_0 - 4k = -3 - 4k$.

**15.** $(b)$ There cannot be any solutions such that $x$ and $y$ are both positive integers, for $x, y \geq 1$ imply $123x + 57y \geq 123 + 57 = 180 > 30$.

## Problems from Burton, p. 237

**1.** $(b)$ Use the hint and part $(a)$. We have a polynomial $x^3 + bx^2 + cx + d$ with a root $r/s$, where $r$ and $s$ are relatively prime integers, and therefore $s$ divides $a = 1$ and $r$ divides $d$. This means that $\pm r$ is a root of the equation, and by the previous sentence we know that $r|d$.■

**2.** $(b)$ By the results of the preceding exercise a rational root must have the form $r/s$ where $r = \pm 1$ and $s$ is equal to $2^m$ for some $m$ satisfying $0 \leq m \leq 5$. In fact, $\frac{1}{2}$ and $-\frac{1}{4}$ are the roots of this polynomial, and the latter has multiplicity 2 (in other words, $(4x+1)^2$ divides the polynomial).■

$(d)$ Every rational root of this monic polynomial must be an integer and a divisors of the constant term 24, and hence the only possibilities are $\pm 1$, $\pm 2$, $\pm 3$, $\pm 4$, $\pm 6$, $\pm 8$, $\pm 12$, and $\pm 24$. Now the polynomial $p(x) = x^3 - 7x^2 + 20x - 24$ is negative for $x < 0$, so we can narrow down the possibilities to the positive divisors of 24. Now $p(x) > 0$ if $x \geq 8$ and this reduces the options to 1, 2, 3, 4 and 6. One can then check directly that 3 is the only root.■

## SOLUTIONS TO ADDITIONAL EXERCISES

**1.** The surface of revolution obtained from the wire is the sphere of radius 1. Therefore the Pappus Centroid Theorem says that the area of the surface of the sphere is the length of the wire times the circumference of a circle of radius $\overline{y}$ where $\overline{y}$ is the $y$-coordinate of the centroid of the wire.

We know that the surface area is $4\pi$ and the length of the wire is $2\pi$, and therefore by the Pappus Centroid Theorem we have

$$4\pi \quad = \quad (2\pi) \cdot (2\pi\,\overline{y})$$

and if we solve this for $\overline{y}$ we obtain $\overline{y} = 1/\pi$.

Similarly, the solid of revolution obtained from the half disk is a sphere of radius 1 and therefore its volume is $\frac{4}{3}\pi$. In this case the Pappus Centroid Theorem says that the volue of the solid sphere is the area of the semicircular disk times the circumference of a circle of radius $\overline{y}$ where $\overline{y}$ is the $y$-coordinate of the centroid of the half disk.

Since the area of the half disk is $\frac{1}{2}\pi$ it follows that

$$\tfrac{4}{3}\pi \quad = \quad (\tfrac{1}{2}\pi) \cdot (2\pi\,\overline{y})$$

and if we solve this for $\overline{y}$ we obtain $\overline{y} = 4/3\,\pi$.

Therefore the inequality

$$\frac{4}{3\,\pi} \quad > \quad \frac{1}{\pi}$$

implies that the centroid of the semicircular wire is closer to the center of the circle than the centroid of the half disk.■

**2.** If we rotate the half ellipse about the $x$-axis we obtain an ellipse whose principal axes have lengths $a$, $b$ and $a$ in the $x$, $y$ and $z$ directions.

Before proceeding we note that the area formula for the ellipse also works if the lengths of the major and minor axes are $b$ and $a$ respectively; in fact it would probably be better to say simply that $a$ and $b$ are supposed to be the lengths of the principal axes.

The area of half the ellipse is $\frac{1}{2}\,\pi\,a\,b$, so by the Pappus Centroid Theorem we have

$$\tfrac{4}{3}\,\pi\,a^2\,b \quad = \quad \left(\tfrac{1}{2}\,\pi\,a\,b\right)\cdot(2\,\pi\,\overline{y})$$

and if one solves this equation one finds that

$$\overline{y} \quad = \quad \frac{4\,a}{3\,\pi}$$

is the $y$-coordinate for the centroid.

**3.** We first need to find the centroid of the triangle. General results, or integral calculus applied to the specific example, imply that the centroid of the triangle in the problem is equal to $(2, c+\frac{4}{3})$. The area of the triangle is equal to $\frac{1}{2}\cdot 3\cdot 4 = 6$, and therefore by Pappus' Theorem the volume for the solid of revolution in the problem is $12\pi(c+\frac{4}{3})^2$.

**4.** The most direct method of this problem is to consider the remainders left by the numbers $(7k+x)^2$ after division by 7, where $x = 1, 2, 3, 4, 5, 6$. Here is what we get:

$(7k+1)^2 = 49k^2 + 14k + 1$ leaves a remainder of 1.
$(7k+2)^2 = 49k^2 + 28k + 4$ leaves a remainder of 4.
$(7k+3)^2 = 49k^2 + 42k + 9$ leaves a remainder of 2.
$(7k+4)^2 = 49k^2 + 56k + 16$ leaves a remainder of 2.
$(7k+5)^2 = 49k^2 + 70k + 25$ leaves a remainder of 4.
$(7k+6)^2 = 49k^2 + 84k + 36$ leaves a remainder of 1.

Therefore the equation $x^2 = 7y + r$ has integral solutions if $r = 1, 2, 4$ but has no solutions if $r = 3, 5, 6$. Furthermore, if $r = 1$ all solutions have the form $7y+1$ or $7y+6$, if if $r = 2$ all solutions have the form $7y+3$ or $7y+4$, and if if $r = 4$ all solutions have the form $7y+2$ or $7y+5$.

**5.** We are going to need the following basic fact: Two lines with equations $p_i x + q_i y = r_i$, where $i = 1$ or 2, are perpendicular if and only if $p_1 q_1 + p_2 q_2 = 0$. — This is true because the directions of the two lines are determined by the vectors $(q_i, -p_i)$, and these two vectors are perpendicular if and only if $p_1 q_1 + p_2 q_2 = 0$.

We can check directly that $(0,0)$ and $(-a,-b)$ are on the line with equation $bx - ay = 0$, and $(0,a)$ and $(b,0)$ are on the line with equation $ax + by = ab$. By the remarks in the first paragraph, these lines are perpendicular since

$$\big(b\cdot a\big) \;+\; \big((-a)\cdot b\big) \quad = \quad 0\;.$$