Math 153
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SOLUTIONS TO EXERCISES FROM `math153exercises05a.pdf`

GENERAL REMARKS ON METHODS. In each of the problems below, we end up with a system of $n-1$ linear equations with integer coefficients in n unknowns, where $n = 2$ or 3. The equations are independent, which means that none have the trivial form $\sum\ 0 \cdot x_i = 0$ (hence all choices of x_i yield solutions) or, when $n = 3$ neither equation is given by multiplying the other by some nonzero constant. In each case, all rational solutions are given by solving for all but one unknowns in terms of the remaining one. After doing this it is necessary to determine when one can find a rational solution for which all the variables are nonnegative integers.

Example 1. Find all nonnegative integral solutions of $6x + 9y = 57$. — We could do this by trial and error, but here is the systematic approach. Solving for one variable in terms of the other, we have

$$x \quad = \quad \frac{57 - 9y}{6} \quad = \quad 9 \ - \ y \ + \ \frac{1}{2}(1 \ - \ y) \ .$$

In order to have an integral solution, the third term must be an integer, which means that $1 - y$ is even or equivalently that y is odd. If we now write $y = 2s + 1$ then this equation becomes $x = (8 - 2s) - s = 8 - 3s$, so the integral solutions are given by letting s be an arbitrary integer and finding x using the formula. If we want a nonnegative integer solution, then we want $s \geq 0$ and $8 - 3s > 0$; the only integers which satisfy these equations are $s = 0, 1, 2$. This means that the solutions are given by $y = 1, 3, 5$ and by substitution the other parts of these solutions are given by $8, 5, 2$.

Example 2. A farmer buys 100 animals for 100 dollars. Cows cost 4 dollars, pigs cost 2 dollars, and chickens cost 50 cents. Find all **positive** integral solutions for this problem. — In this case we obtain the system $4x + 2y + \frac{1}{2}z = 100$ and $x + y + z = 100$ with x, y, z all positive integers. The first step is to eliminate z and obtain an equation in x and y. If we subtract the second equation from twice the first, we obtain $7x + 3y = 100$. As in the first example, we can solve this for y in terms of x, obtaining

$$y \quad = \quad \frac{100 - 7x}{3} \quad = \quad 33 \ - \ 2x \ + \ \frac{1}{3}(1 - x)$$

which means that $x - 1$ is divisible by 3 or $x = 3s + 1$ for some $s \geq 0$. Therefore we must have $y = 33 - 2 - 6s - s = 31 - 7s$. Since $y > 0$ we must also have $s < 5$. So there are **five** possible solutions with $x = 1, 4, 7, 10, 13$ and $y = 31, 24, 17, 10, 3$. It follows that $z = 68, 72, 76, 80, 84$; note that the first equation implies that z must be even.

1. There are four solutions, with 2, 5, 8, 11 horses.

2. The only solution has 3 of each.

3. There is no solution.

4. There are six solutions, with $(H, C) = (2, 26),\ (5, 21),\ (8, 16),\ (11, 11),\ (14, 6),\ (17, 1)$.

5. There are no solutions.

6. There is one solution, with 9 half dollars, 4 quarters, 5 dimes.