Math 153
Spring 2010
R. Schultz

SOLUTIONS TO EXERCISES FROM `math153exercises09.pdf`

As usual, "Burton" refers to the Seventh Edition of the course text by Burton (the page numbers for the Sixth Edition may be off slightly).

Problems from Burton, p. 326

1. (*b*) Set $y = x + 2$, so that

$$2 \quad = \quad x^3 + 6x^2 + 3x \quad = \quad (x+2)^3 - 12x - 8 + 3x \quad =$$

$$(x+2)^3 - 9(x+2) + 18 - 8 = y^3 - 9y + 10$$

and hence the original polynomial reduces to $y^3 - 9y + 8 = 0$. One root of this polynomial is $y = 1$, and $y^3 - 9y + 8 = (y-1)(y^2 + y - 8)$, so the remaining roots are $-\frac{1}{2}(1 \pm \sqrt{33})$. Using the inverse substitution $x = y - 2$ we see that the roots of the original polynomial are -1 and $-\frac{1}{2}(5 \pm \sqrt{33})$.■

(*d*) We may rewrite the equation as $x^3 - 6x^2 - 24x + 64 = 0$. Now set $y = x - 2$, so that

$$0 \quad = \quad x^3 \ - \ 6x^2 \ - \ 24x \ + \ 64 \quad = \quad (x-2)^3 \ - \ 12x \ + \ 8 \ - \ 24x \ + \ 64 \quad =$$

$$(x-2)^3 \ - \ 32(x-2) \quad = \quad y^3 \ - \ 32y \ .$$

It follows that $0 = y(y^2 - 32)$, so that $y = 0$ or $\pm 4\sqrt{2}$ and $x = y + 2 = 2$ or $2 \pm 4\sqrt{2}$.

3. Before working these exercises, we shall describe one method for computing the coefficients of $y^3 + py + q$ if we start with $f(x) = x^3 - bx^2 + cx + d$ and set $y = x - \frac{1}{3}b$ to eliminate the second degree term.

CLAIM: $q = f(\frac{1}{3}b)$ and $p = f'(\frac{1}{3}b)$.

These follow from the Taylor polynomial expansion of $f(x)$ as a polynomial in $(x - r)$, where $r = \frac{1}{3}b$:

$$f(x) \quad = \quad f(r) \ + \ f'(r)(x-r) \ + \ \frac{f''(r)}{2}(x-r)^2 \ + \ \frac{f'''(r)}{6}(x-r)^3$$

At the very least this is a good check on the algebraic i computations if one simply makes the substitution $x = y + \frac{1}{3}b$ and writes everything out as a polynomial in y.

(*b*) We shall try to use Cardan's formula as printed on page 326. But first we need to rewrite the given equation as $y^3 + py + q = 0$. We are given $0 = x^3 - 6x^2 + 15x - 18$, and if we set $y = x - 2$ this becomes $y^3 + 3y - 4 = 0$, so that $p = 3$ and $q = -4$. Thus

$$\frac{q}{2} \quad = \quad -2 \quad , \quad \frac{q^2}{4} \quad = \quad 4 \quad , \quad \frac{p^3}{27} \quad = \quad 1$$

, so the formula yields the root value

$$\sqrt[3]{-2 + \sqrt{4-1}} \ + \ \sqrt[3]{-2 - \sqrt{4-1}} \quad = \quad \sqrt[3]{-2 + \sqrt{3}} \ + \ \sqrt[3]{-2 - \sqrt{3}} \ .$$

(d) We shall try to use Cardan's formula as printed on page 326. In our situation $p = 9$ and $q = 12$. Thus

$$\frac{q}{2} \;=\; 6 \;\;,\;\; \frac{q^2}{4} \;=\; 36 \;\;,\;\; \frac{p^3}{27} \;=\; 27$$

, so the formula yields the root value

$$\sqrt[3]{6 + \sqrt{36 - 27}} \;+\; \sqrt[3]{6 - \sqrt{36 - 27}} \;\;=\;\; \sqrt[3]{6 + \sqrt{9}} \;+\; \sqrt[3]{6 - \sqrt{9}} \;\;=$$

$$\sqrt[3]{6 + 3} \;+\; \sqrt[3]{6 - 3} \;\;=\;\; \sqrt[3]{9} \;+\; \sqrt[3]{3}\;.$$

(f) In this case the polynomial can be rewritten in the form $x^3 - 3x^2 - 27x + 41 = 0$, and if we set $y = x - 1$ this transforms to $y^3 - 30y + 12 = 0$, so that $p = -30$ and $q = 12$. Thus

$$\frac{q}{2} \;=\; 6 \;\;,\;\; \frac{q^2}{4} \;=\; 36 \;\;,\;\; \frac{p^3}{27} \;=\; -1000$$

so the formula yields the root value

$$\sqrt[3]{6 + \sqrt{36 + 1000}} \;+\; \sqrt[3]{6 - \sqrt{36 + 1000}} \;\;=\;\; \sqrt[3]{6 + \sqrt{1036}} \;+\; \sqrt[3]{6 - \sqrt{1036}}\;.$$

SOLUTIONS TO ADDITIONAL EXERCISES

1. (a) We should begin by observing that every monic cubic polynomial with real coefficients factors into a product of three monic first degree polynomials over the complex numbers; this is actually a special case of the so - called **Fundamental Theorem of Algebra**, which states that every polynomial of positive degree with complex coefficients factors completely into a product first degree polynomials, but we do not need the full force of this result (which will be discussed further in Unit 13). — To verify the factorization for cubic polynomials, note first that if $f(x)$ is a monic cubic polynomial with real coefficients then $\lim_{x \to \pm\infty} f(x) = \pm\infty$, so that there is some $M > 0$ for which $f(x)$ is negative when $x < -M$ and $f(x)$ is positive when $x > M$. By the Intermediate Value Theorem from calculus, it follows that $f(x)$ has at least one real root, which we shall call a. Therefore by the Factor Theorem for polynomials $f(x)$ can be factored as $(x-a)\,g(x)$ where $g(x)$ is a quadratic polynomial with real coefficients. By the Quadratic Formula and the Factor Theorem we can now write $g(x) = (x - b)(x - c)$, where b, c are complex numbers. If we combine these, we obtain the factorization $f(x) = (x - a)(x - b)(x - c)$.

Given the factorization $f(x) = (x-a)(x-b)(x-c)$ in the preceding sentence, we can compute directly that

$$f(x) \;\;=\;\; x^3 \;-\; (a+b+c)x^2 \;+\; (\text{linear term})$$

and therefore $a + b + c = 0$ if and only if the coefficient of the second degree term is zero.

(b) Rewrite the cubic equation in the form $x^3 - cx + d = 0$. Now r and s are known to be positive roots of this equation, and if t is the third root (counting multiple roots the appropriate number of times) then by the first part of this exercise the sum $r + s + t$ must be zero, so that $t = -(r + s)$. If we substitute this into the original equation we see that

$$0 \;\;=\;\; \big(-(r+s)\big)^3 \;-\; \big(-(r+s)\big) \;+\; d \;\;=\;\; -(r+s)^3 \;+\; (r+s) \;+\; d$$

which means that $r+s$ is a root of $x^3 - cx - d = 0$, or equivalently that $r+s$ solves the equation $x^3 = cx + d$.

2. As in the preceding problem if a is a root of $x^3 - cx - d = 0$, then $-a$ is a root of $x^3 - cx + d = 0$, for if we evaluate the latter at $-a$, we obtain $-a^3 + ca + d = -(a^3 - ca - d) = -0 = 0$. This leads to the following factorization:

$$x^3 - cx + d \quad = \quad (x+a)\left(x^2 \; - \; \frac{ac+d}{a^2}\,x \; + \; \frac{d}{a}\right)$$

If we use the Quadratic Formula to find the roots of the second factor and make substitutions using the equation $a^3 = ca + d$, we obtain the expressions displayed in the exercise.

To solve the equation $x^3 + 3 = 8x$ using the preceding, we need to begin by finding a root for $x^3 = 8x + 3$. Fortunately, we can see (say by trial and error) that $x = 3$ solves the second equation. If we now substitute the values $a = 3$, $c = 8$ and $d = 3$ into the formula, we see that one root of the original equation is -3 and the other roots of the original equation are $\frac{1}{2}(3 \pm \sqrt{5})$.

3. Follow the hint and find the critical points of the polynomial $p(x) = x^3 - cx - d$. If $0 = p'(x) = 3x^2 - c$ then it follows that the critical points are $x = \pm\sqrt{c/3}$, and by the Second Derivative test the negative point x_M is a relative maximum while the positive point is a relative minimum. As indicated in the hint, we know that $p(0) = -d < 0$, and since

$$\lim_{x\to+\infty} p(x) \quad \lim_{x\to+\infty} x^3 \cdot \left(1 \; - \; \frac{c}{x^2} \; - \; \frac{d}{x^3}\right) \quad = \quad \lim_{x\to+\infty} x^3 \quad = \quad +\infty$$

it follows from the Intermediate Value Theorem that $p(x)$ has at least one positive real root.

To see that $p(x)$ has two negative real roots, it is enough to show that $p(x_M) > 0$, for then we can reason as in the previous paragraph to conclude that $\lim_{x\to-\infty} p(x) = -\infty$, so that p as one root r_1 satisfying $r_1 < x_M$ and another root r_2 satisfying $x_M < r_2 < 0$. But we have

$$p(x_M) \quad = \quad p\left(-\sqrt{\frac{c}{3}}\right) \quad =$$

$$-\left(\frac{c}{3}\right)^{3/2} \; + \; c \cdot \left(\frac{c}{3}\right)^{1/2} \; - \; d \quad = \quad \frac{2c}{3} \cdot \left(\frac{c}{3}\right)^{1/2} \; - \; d$$

which is positive if and only if the square of the first term is greater than the square of the second. But the latter condition is precisely the assumption

$$\left(\frac{c}{3}\right)^3 \quad > \quad \left(\frac{d}{2}\right)^2$$

in the exercise.

4. The derivative of $p(x) = x^3 + cx - d$ is $3x^2 + c$, which is positive for all values of x. Therefore it follows that the value of the polynomial is negative. If $p(x)$ had some negative root, then by the Mean Value Theorem implies that $p'(y)$ would have to be negative for some $y < 0$. Therefore p has no negative roots. On the other hand, since p is strictly increasing and $\lim_{x\to\infty} p(x) = +\infty$, it follows that p must have at least one positive root. But since p is always positive it cannot have more than one such root (otherwise $p'(y) = 0$ for some $y > 0$).

5. In this problem, it is much better if one disregards the hint and looks at the equation $\cos 3\theta = \cos 120^\circ = -\frac{1}{2}$. Then as in `history02.pdf` we see that $-\frac{1}{2} = 4y^3 - 3y$, so that $8y^3 - 6y + 1 = 0$.

Note. We can check that this is not rational by using the methods of `history02.pdf` to check that the polynomial $z^3 - 3z + 1$ has no rational roots; as in that document, r is a root of this polynomial if and only if $2r$ is a root of the polynomial for $\cos 40^\circ$, and once again by Gauss' result the only possible rational roots for the latter would be ± 1, which cannot happen because $(\pm 1)^3 - 3(\pm 1) + 1$ must be an odd number.

6. We know that

$$\cos 22\tfrac{1}{2}^\circ \quad = \quad \sqrt{\frac{1}{2} + \frac{\sqrt{2}}{4}}\,, \quad \sin 22\tfrac{1}{2}^\circ \quad = \quad \sqrt{\frac{1}{2} - \frac{\sqrt{2}}{4}}$$

and if we square these we see that the sine and cosine of $22\frac{1}{2}^\circ$ satisfy the quadratic equations

$$x^2 \quad = \quad \frac{1}{2} \pm \frac{\sqrt{2}}{4}\,.$$

Therefore it will suffice to find a quadratic polynomial which has both of the two numbers on the right as roots. But one can do this easily, for the quadratic polynomial must be $y^2 - y + (1/8) = 0$, so that the original numbers are roots of $x^4 - x^2 + (1/8) = 0$.

7. This uses the same idea as the preceding exercise. The number $3 - \sqrt{2}$ is a root of the polynomial equation $y^2 - 6y + 5 = 0$, so if $x^2 = 3 - \sqrt{2}$ then $x^4 - 6x^2 - 5 = 0$.