NAME: ______________________________

Mathematics 153–001, Spring 2012, Examination 3

Answer Key

1. [*20 points*] Given the cubic equation $x^3 + bx^2 + d = 0$, find a number k such that the change of variables $x = y - k$ yields an equation of the form $y^3 + py + q = 0$, and describe the coefficients p and q in terms of b and d.

SOLUTION

If $x = y - k$ then we have

$$0 \;=\; (y-k)^3 \;+\; b(y-k)^2 \;+\; d \;=\; y^3 \;+\; (b-3k)y^2 \;+\; (3k^2-2bk)y \;+\; (d+bk^2-k^3)\;.$$

Since the coefficient of y^2 is zero this means that $k = b/3$, and if we substitute this into the coefficients of y and 1 (*i.e.*, the constant term), we see that

$$p \;=\; \frac{b^2}{9} - \frac{2b^3}{3} \;=\; -\frac{5b^2}{9}\;, \qquad q \;=\; d \;+\; \frac{b^3}{9} - \frac{b^3}{27} \;=\; d \;+\; \frac{2b^3}{27}\;.$$

2. [*15 points*] Find real numbers a and b such that the complex number

$$\frac{2+3i}{2-3i}$$

is equal to $a+bi$.

SOLUTION

We have

$$\frac{1}{2-3i} \quad = \quad \frac{2+3i}{13 = 2^2+3^2}$$

and hence the number in question is equal to

$$\frac{(2+3i)^2}{13 = 2^2+3^2} \quad = \quad \frac{-5+12i}{13} \ .$$

3. [*20 points*] (*a*) Given that $\log_{10} 2 \approx 0.3010$, find the number of digits in the integer 2^{256}.

(*b*) Evaluate the infinite sum

$$\sum_{k=1}^{\infty} \frac{k(k-1)}{3^k} \; .$$

[*Hint:* Differentiate the geometric series $1/(1-x) = \Sigma\, x^k$ twice and then multiply by x^2.]

SOLUTION

(*a*) We have $\log_{10} 2^{256} \;=\; 256 \cdot \log_{10} 2 \;\approx\; 256 \cdot 0.3010 \;=\; 77.056$, and hence the number of digits in 2^{256} is equal to

$$1 \;+\; \left[\log_{10} 2^{256}\right] \;=\; 1 \;+\; 77 \;=\; 78 \; .$$

(*b*) Follow the hint:

$$\frac{d^2}{dx^2}\,\frac{1}{1-x} \;=\; \frac{2}{(1-x)^3}\;, \qquad \frac{d^2}{dx^2}\sum_k x^k \;=\; \sum_k k(k-1)x^{k-2}$$

so that the right hand sides are equal, and if we multiply both sides by x^2 we obtain the following identity:

$$\frac{2x^2}{(1-x)^3} \;=\; \sum_k k(k-1)x^k$$

We want to evaluate the right hand side when $x = 1/3$, and we can do this by setting $x = 1/3$ in the left hand side. If we do so, we get

$$\sum_{k=1}^{\infty} \frac{k(k-1)}{3^k} \;=\; \frac{2/9}{(2/3)^3} \;=\; \frac{2\cdot 27}{9\cdot 8} \;=\; \frac{3}{4}\; .$$

4. [*15 points*] Given a rectangle, let L and W denote its length and width, so that its perimeter $P = 2L+2W$. Prove that a square has the greatest area among all rectangles with a fixed perimeter P.

SOLUTION

We know that the area is $A = LW$, and we also know that

$$W = \frac{P-2L}{2}$$

so the problem is to find L such that A is maximized. Since $A(L) = \frac{1}{2}L(P-2L)$, we have

$$A'(L) = \frac{P-4L}{2}$$

and the maximum will occur when this is zero. The formula shows that $A'(L) = 0$ if and only if $P = 4L$, and since $P = 2L + 2W$ this happens if and only if $L = W$, which is true if and only if the rectangle is a square.

5. [*15 points*] Answer the following questions. BRIEF statements of reasons may be included and may earn partial credit if answers are incorrect.

ANSWERS ARE GIVEN IN CAPITAL LETTERS.

(a) For each of the following features of calculus, state whether it is due to Leibniz, Newton, both or neither.

Who developed the infinite series expansion for $\sqrt[n]{1+x}$ when $|x| < 1$? NEWTON
Who used notation like $\int f(x)\,dx$ for integrals? LEIBNIZ
Who recognized the fact that differentiation and integration were essentially inverse process (differentiation undoes integration and vice versa)? BOTH

(b) Given two names from the list below who contributed to the development of analytic (coordinate) geometry. Extra credit is possible for more than two names but will be offset by penalties for incorrect additional names.

Brouncker
Clairaut YES
Fermat YES
La Hire YES
La Roche
Mersenne
Pacioli
Riese
Schooten YES
Stevin
Viète YES

6. [*20 points*]

ANSWERS ARE GIVEN IN CAPITAL LETTERS.

(*a*) Which of the following statements is correct?

Formulas for the roots of cubic and quartic equations were found in the 16^{th} century, and a formula for the roots of quintic equations was found in the 18^{th} and 19^{th} centuries. FALSE

A formula for the roots of cubic equations was found in the 16^{th} century, and formulas for the roots of quartic and quintic equations were found in the 18^{th} and 19^{th} centuries. FALSE

Formulas for the roots of cubic and quartic equations were found in the 16^{th} century, and the nontexistence of a formula for the roots of quintic equations was established in the 18^{th} and 19^{th} centuries. TRUE

A formula for the roots of cubic equations was found in the 16^{th} century, and the nonexistence of formulas for the roots of quartic and quintic equations was established in the 18^{th} and 19^{th} centuries. FALSE

(*b*) Which of the following contributed to calculus before Newton and Lebniz, and which of the following contributed to calculus after Newton and Lebniz? For full credit, answer this for five of the names listed below. Extra credit is possible for more than five names but will be offset by penalties for incorrect additional names.

Barrow BEFORE
Jacques Bernoulli AFTER
Bolzano AFTER
Cauchy AFTER
Cavalieri BEFORE
Dirichlet AFTER
Euler AFTER
Gregory BEFORE
Lagrange AFTER
Pascal BEFORE
Torricelli BEFORE
Weierstrass AFTER

Additional sheets for use if needed.

Second additional sheet.