14.F. Uniform convergence of Fourier series

The discussion below is based upon Rudin's *Principles of Mathematical Analysis* (Third Ed.), and particularly on its treatment of Fourier series.

If f is a periodic function of bounded variation on the closed interval $[0, 2\pi]$ (with period 2π), then by Dirichlet's Theorem we know that the Fourier series of f converges to f at all but countably many points at which f is discontinuous and that it converges to $\frac{1}{2}\big(f(x+) + f(x-)\big)$, where $f(x\pm)$ denotes the appropriate one-sided limit of $f(t)$ as $t \to x$ from the indicated side. Since the uniform limit of continuous functions is continuous, it follows immediately that the convergence of this series will not be uniform if f is not continuous. However, there is a simple condition under which the Fourier series does converge uniformly to the original function.

THEOREM. *Suppose that f is a periodic function of period 2π with a continuous derivative everywhere. Then the Fourier series for f converges to f uniformly.*

Proof. Given a continuous periodic function g of period 2π, let $a_n(g)$ and $b_n(g)$ denote the coefficients of its Fourier series. Then integration by parts yields

$$a_n(f') = nb_n(f) , \qquad b_n(f') = -na_n(f)$$

if $n > 0$, and since f and f' are both periodic integration by parts also yields $a_0(f') = 0$. Since the sum of the squares of the Fourier series coefficients is convergent by Bessel's Inequality, it follows that

$$S_2(f') = \sum_{n=1}^{\infty} n^2 a_n^2(f) + n^2 b_n^2(f) < \infty .$$

By the "infinite series" version of the Cauchy-Schwarz Inequality it then follows that

$$\sum_{n=1}^{\infty} \sqrt{a_n^2(f) + b_n^2(f)} \leq S_2(f') \cdot \sum_{n=1}^{\infty} \frac{1}{n^2}$$

and hence the left hand side converges. If we combine this with the inequality $|\alpha| + |\beta| \leq 2\sqrt{\alpha^2 + \beta^2}$ and the comparison test for convergence of infinite series, we see that the series

$$S_2(f) = \frac{|a_0(f)|}{2} + \sum_{n=1}^{\infty} |a_n(f)| + |b_n(f)|$$

converges. Therefore by the Weierstrass text for uniform convergence it follows that the Fourier series of f, which is

$$\frac{a_0(f)}{2} + \sum_{n=1}^{\infty} a_n(f) \cos nx + b_n(f) \sin nx$$

must converge uniformly to some function g. Since f is a function of bounded variation (continuous derivative), it follows that the Fourier series converges pointwise to f, and therefore the uniform limit of the Fourier series must be the original function f.