1.B. Base 60 expansions of unit fractions

One can compute the Babylonian expansions for unit fractions of the form $1/n$ in very much the same way as one computes decimal expansions. We shall begin by reviewing the process for base 10. A proof that this gives the decimal expansion for $1/n$ is given on page 105 of the following online document:

http://math.ucr.edu/~res/math144/setsnotes5.pdf

The coefficients a_n in the decimal expansion

$$\frac{1}{n} \quad = \quad 0.a_1 a_2 a_3 \cdots$$

nay be determined recursively as follows: By long division we may write

$$10 \quad = \quad a_1\, n + b_1$$

where b_1 is an integer satisfying $0 \leq b_1 < n$. Note that since n is a positive integer it follows that $a_1 \leq a_1\, n \leq 10$. To find the next coefficient we proceed as in long division, bringing down the term b_1 and finding a_2 and b_2 such that

$$10\, b_1 \quad = \quad a_2\, n + b_2$$

where now b_2 is an integer satisfying $0 \leq b_2 < n$. Once again we have

$$a_2\, n \quad \leq \quad 10\, b_1 \quad < \quad 10\, n$$

which implies that $0 \leq a_2 \ < 10$. Using b_2 we may find a_3 and b_3 similarly, and so forth. If n evenly divides some power of 10, then eventually one obtains a remainder $b_k = 0$, and at that point all subsequent terms a_j and b_j will also be equal to zero.

The same procedure works for other bases, and in particular for base 60. Let's see exactly what happens if we apply it to a simple example:

Problem. Express 1/48 in Babylonian-type notation.

SOLUTION. In this case the first step is

$$60 \quad = \quad 48\, a_1 \ + \ b_1$$

and the conditions imply that $a_1 = 1$ and $b_1 = 12$. At the next stage we have

$$720 \quad = \quad 60 \cdot 12 \quad = \quad 48 \cdot 15 \ + \ 0 \ .$$

This means that the first term in the expansion is 1 and the second is 15, or in the modern representation we have the sexagesimal fraction 0;1,15 as our answer.

Let's verify this:

$$\frac{1}{60} + \frac{15}{3600} \quad = \quad \frac{75}{3600} \quad = \quad \frac{1}{48}$$

In analogy with the base 10 case, this process always terminates after a finite number of steps if n evenly divides some power or 60, or equivalently if the only prime divisors of n are contained in the set $\{2, 3, 5\}$.

Another example. What is the expansion for 1/1000?

ANSWER. In this case the algorithm yields the expression 0;0,3,36 for the fraction in question. At the first step one obtains $a_1 = 0$ and $b_1 = 60$, while at the second step one obtains $a_2 = 3$ and $b_2 = 600$, and at the third step one obtains $a_3 = 36$ and $b_3 = 0$.

Eventual periodicity of base 60 *expansions*

The results on page 105 of the document

`http://math.ucr.edu/~res/math144/setsnotes5.pdf`

also show that if $0 < p < q$ then the coefficients a_k in the decimal expansion

$$\frac{p}{q} \quad = \quad 0.a_1 a_2 a_3 \cdots$$

are *eventually periodic* in the sense that one can find positive integers m and Q such that $k \geq m$ implies $a_k = a_{k+Q}$, and conversely if we are given integral coefficients a_j which are eventually periodic and satisfy $0 \leq a_j < 10$, then the decimal expansion

$$\sum_{j \geq 1} \frac{a_j}{10^j}$$

is a rational number. The argument generalizes to base N expansions in which 10 is replaced by an arbitrary integer $N > 1$ (as do all the proofs on pages 103–109 of that document). In particular, this is true for $N = 60$. However, as indicated in `history01.pdf` all base 60 expansions in Babylonian mathematics were truncated at the third term and hence such patterns were most likely of no interest at the time.