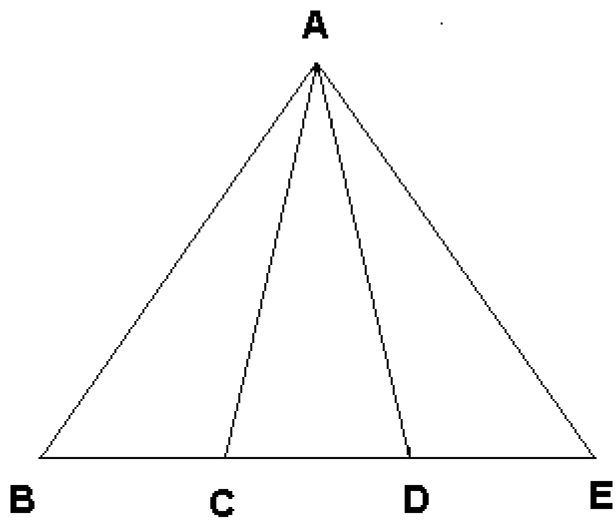
2.C. An Easy Trisection Fallacy

We have already noted that any purported straightedge and compass construction for trisecting an angle will be incorrect. The following simple example illustrates how appealing such a construction might appear at first and how one can look more closely to find a mistake.

Suppose we are given an angle $\angle$ **BAE** as in the diagram below and we wish to trisect it. Let's assume that the lengths of the segments **[BA]** and **[AE]** are equal. It is known that segments can be divided into any number of pieces of equal length by straightedge and compass, so apply this to segment **[BE]** and divide it into three equal segments that we shall call **[BC], [CD]** and **[DE].** If we look at the picture it might seem that the rays **[AC** and **[AD** trisect $\angle$**BAE,** but is this really true**?**



One can use the classical methods of Euclidean geometry to conclude that the segments **[AC]** and **[AD]** have equal length, and it is possible to analyze this figure even further using classical methods, but we shall take a shortcut using trigonometry.

Let h denote the common altitude of the isosceles triangles $\triangle$**BAE** and $\triangle$**CAD,** and let **|XY|** denote the length of the segment joining **X** and **Y.** Then standard results in trigonometry imply the following relationships:

$$\tan \tfrac{1}{2}\angle \mathrm{BAE} = |\mathrm{BE}|/2h \qquad\qquad \tan \tfrac{1}{2}\angle \mathrm{CAD} = |\mathrm{CD}|/2h = |\mathrm{BE}|/6h$$

From these formulas we conclude that $\tan \tfrac{1}{2}\angle \mathrm{CAD}$ is one third of $\tan \tfrac{1}{2}\angle \mathrm{BAE}$. If this construction yielded a trisection then we would have a trigonometric identity of the form

$$(\tan x)/3 = \tan (x/3)$$

and one can check directly from tables (or a scientific calculator) that the first expression is always greater than the second. Since the tangent function is strictly increasing, it follows that ***the middle angle is always larger than the angles on both sides*****.**

It is also possible to disprove this trisection fallacy using classical methods from Euclidean geometry, but the argument is somewhat longer.