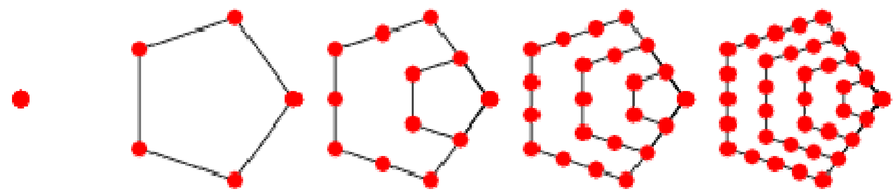
2.E. Polygonal numbers

The Pythagoreans were interested in certain geometrically determined sequences of numbers called polygonal numbers, and the first few cases (***triangular*** and ***square*** numbers) are mentioned on page 95 of Burton. Pentagonal numbers are also mentioned; their definition is suggested by the following picture.



(Source: http://mathworld.wolfram.com/PentagonalNumber.html)

As in the case of triangular and square numbers, if one knows the n^{th} pentagnal number p_n then the next one is given recursively in terms of p_n. One of the exercises for this unit is to find the recursive formula and to derive a closed formula for p_n as an explicit function of n.

Clearly one can proceed indefinitely, starting with hexagonal numbers. The following online references contain further information on this topic:

http://en.wikipedia.org/wiki/Polygonal_number

http://mathworld.wolfram.com/PolygonalNumber.html

One particularly noteworthy result about polygonal numbers is the ***Polygonal Number Theorem***, which was conjectured by P. de Fermat in the 17^{th} century. This result states that, for all $m \geq 3$, every positive integer is a sum of m numbers that are m – gonal numbers (a sum of three triangular numbers, four perfect squares, five pentagonal numbers, and so on). Partial results were obtained by several leading mathematicians during the 18^{th} and early 19^{th} centuries, and ultimately the full result was proved by A. L. Cauchy (1789 – 1857). There is a short proof of this result (with a few background references) in the following paper:

Nathanson, Melvyn B. A short proof of Cauchy's polygonal number theorem. *Proc. Amer. Math. Soc.* **99** (1987), 22 – 24.