4. Alexandrian mathematics after Euclid — II

Due to the length of this unit, it has been split into three parts.

Apollonius of Perga

If one initiates a **Google** search of the Internet for the name "Apollonius," it becomes clear very quickly that many important contributors to Greek knowledge shared this name (reportedtly there are 193 different persons of this name cited in Pauly-Wisowa, *Real-Enzyklopädie der klassischen Altertumswissenschaft*), and therefore one must pay particular attention to the full name in this case. The MacTutor article on Apollonius of Perga lists several of the more prominent Greek scholars with the same name.

Apollonius of Perga made numerous contributions to mathematics (Perga was a city on the southwest/south central coast of Asia Minor). As is usual for the period, many of his writings are now lost, but it is clear that his single most important achievement was an eight book work on conic sections, which begins with a general treatment of such curves and later goes very deeply into some of their properties. His work was extremely influential; in particular, efforts to analyze his results played a very important role in the development of analytic geometry and calculus during the 17$^{\text{th}}$ century.

Background discussion of conics

We know that students from Plato's Academy began studying conics during the 4$^{\text{th}}$ century B.C.E., and one early achievement in the area was the use of intersecting parabolas by Manaechmus (380 – 320 B.C.E., the brother of Dinostratus, who was mentioned in an earlier unit) to duplicate a cube (recall Exercise 4 on page 128 of Burton). One of Euclid's lost works was devoted to conics, and at least one other early text for the subject was written by Aristaeus the Elder (*c.* 360 – 300 B.C.E.). Apollonius' work, ***On conics***, begins with an organized summary of earlier work, fills in numerous points apparently left open by his predecessors, and ultimately treat entirely new classes of problems in an extremely original, effective and thorough manner. In several respects the work of Apollonius anticipates the development in coordinate geometry and uses of the latter with calculus to study highly detailed properties of plane curves. Of the eight books on conics that Apollonius wrote, the first four have survived in Greek, while Books **V** through **VII** only survived in Arabic translations and the final Book **VIII** is lost; there have been attempts to reconstruct the latter based upon commentaries of other Greek mathematicians, most notably by E. Halley (1656 – 1742, better known for his work in astromony), but they all involve significant amounts of speculation. Some available evidence suggests that the names ***ellipse***, ***parabola*** and ***hyperbola*** are all due to Apollonius, but the opinions of the experts on this are not unanimous.

Today we think of conics in the coordinate plane as curves defined by quadratic equations in two variables. As the name indicates, ancient Greek mathematicians viewed such curves as the common points of a cone and a plane. One important feature of Apollonius' writings on conics is that they present an integrated framework for analyzing such curves intrinsically in the planes containing them (as opposed to viewing them as intersections of planes and cones). For the sake of completeness, we shall include a summary of this relationship taken from the following two sources:

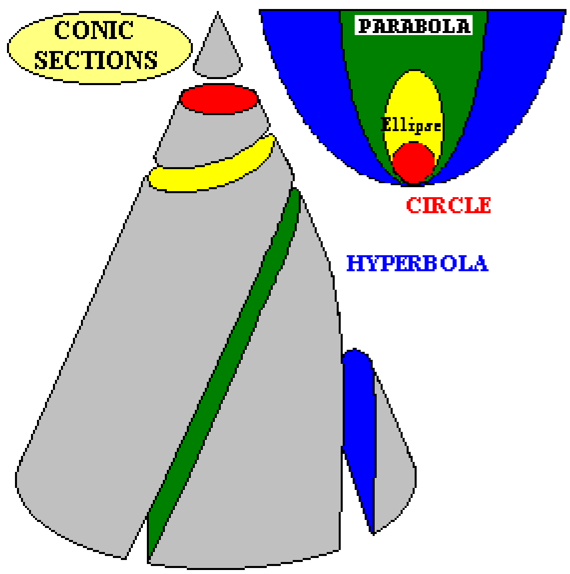
http://www.chemistrydaily.com/chemistry/Conic_section

http://en.wikipedia.org/wiki/Conic_section

Since we are including a fairly long discussion of conics, it seems worthwhile to include a link to a site describing how conics arise in nature and elsewhere:

http://ccins.camosun.bc.ca/~jbritton/jbconics.htm

In the figure below there is a picture from the above references to illustrate how the different conic sections are formed by intersecting a cone with a plane.



(The material which follows is taken from the sites listed above.)

Types of conics

Two well-known conics are the circle and the ellipse. These arise when the intersection of cone and plane is a closed curve. The circle is a special case of the ellipse in which the plane is perpendicular to the axis of the cone. If the plane is parallel to a generator line of the cone, the conic is called a parabola. Finally, if the intersection is an open curve and the plane is not parallel to a generator line of the cone, the figure is a hyperbola. (In this case the plane will intersect ***both*** halves of the cone, producing two separate curves, though often one is ignored.)

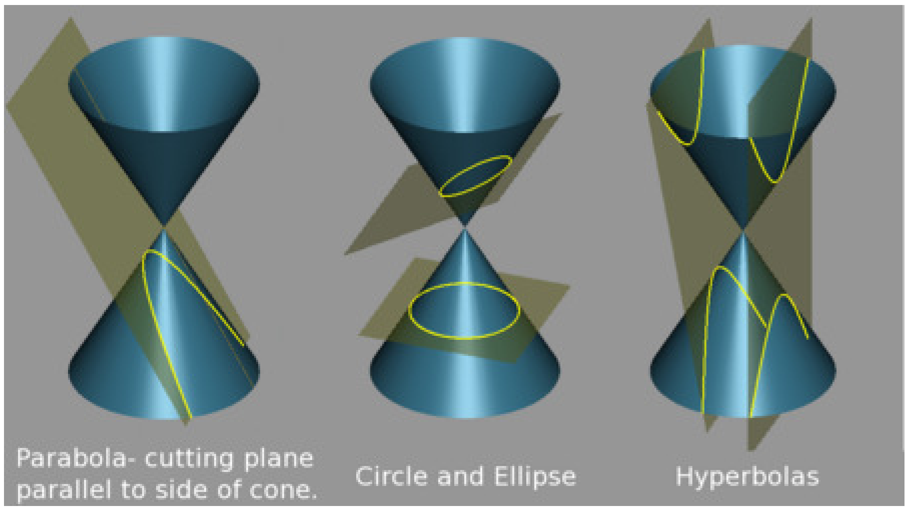
The degenerate cases, where the plane passes through the apex of the cone, resulting in an intersection figure of a point, a straight line or a pair of lines, are often excluded from the list of conic sections.

In Cartesian coordinates, the graph of a quadratic equation in two variables is always a conic section, and all conic sections arise in this way. If the equation is of the form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

then we can classify the conics using the coefficients as follows:

- If $h^2 = ab$, the equation represents a ***parabola***.
- If $h^2 < ab$, the equation represents an ***ellipse***.
- If $h^2 > ab$, the equation represents a ***hyperbola***.
- If $a = b$ and $h = 0$, the equation represents a ***circle***.
- If $a + b = 0$, the equation represents a ***rectangular hyperbola***.



Graphic visualizations of the conic sections

Eccentricity

There are several alternative definitions of non – circular conic sections which do not involve coordinates. One unified approach starts with a point **F** (the ***focus***), a line **L** not containing **F** (the ***directrix***) and a positive number e (the ***eccentricity***); focal points for conics appear in Apollonius' writings, and the concept of directrix for arbitrary conics is apparently due to Pappus of Alexandria (whom we shall discuss in the next unit). The conic section Γ associated to **F**, **L** and e then consists of all points **P** whose distance to **F** equals e times their distance to **L**. When $0 < e < 1$ we obtain an ***ellipse***, when $e = 1$ we obtain a ***parabola***, and when $e > 1$ we obtain a ***hyperbola***.

For an ellipse and a hyperbola, two focus – directrix combinations can be taken, each giving the same full ellipse or hyperbola; in particular, ellipses and hyperbolas have two focal points. The distance from the center of such a curve to the directrix equals a/e, where a is the semi – major axis of the ellipse (the maximum distance from a point on the ellipse to its center), or the distance from the center to either vertex of the hyperbola (the minimum distance from a point on the hyperbola to its center).

In the case of a circle one often takes $e = 0$ and imagines the directrix to be infinitely far removed from the center (for those familiar with the language of projective geometry, the directrix is taken to be the "line at infinity"). However, the statement that the circle consists of all points whose distance is e times the distance to **L** is not useful in such a setting, for the product of these two numbers is formally given by zero times infinity. In any case, we can say that the eccentricity of a conic section is thus a measure of how far it deviates from being circular.

For a given choice of a, the closer e is to 1, the smaller is the semi – minor axis.

Derivations of the conic section equations

Take a cone whose axis is the z – axis and whose vertex is the origin. The equation for the cone is

$$x^2 + y^2 - a^2 z^2 = 0 \qquad (1)$$

where

and θ is the angle which the generators of the cone make with respect to the axis. Notice that this cone is actually a pair of cones, with one cone standing upside down on the vertex of the other cone — or, as mathematicians say, it consists of two "nappes" (pronounced NAPS).

Now take a plane with a slope running along the x direction but which is level along the y direction. Its equation is

$$z = mx + b \qquad (2)$$

where

$$m = \tan\phi > 0$$

and ϕ is the angle of the plane with respect to the xy – plane.

We are interested in finding the intersection of the cone and the plane, which means that equations **(1)** and **(2)** should be combined. Both equations can be solved for z and then equate the two values of z. Solving equation **(1)** for z yields

$$z = \sqrt{\frac{x^2 + y^2}{a^2}}$$

and therefore

$$\sqrt{\frac{x^2 + y^2}{a^2}} = mx + b.$$

Square both sides and expand the squared binomial on the right side:

$$\frac{x^2 + y^2}{a^2} = m^2 x^2 + 2mbx + b^2.$$

Grouping by variables yields

$$x^2\left(\frac{1}{a^2} - m^2\right) + \frac{y^2}{a^2} - 2mbx - b^2 = 0. \qquad (3)$$

Note that this is the equation of the projection of the conic section on the xy – plane, hence contracted in the x direction compared with the shape of the conic section itself.

Equation of the parabola

The parabola is obtained when the slope of the plane is equal to the slope of the generators of the cone. When these two slopes are equal, then the angles θ and ϕ become complementary. This implies that

$$\tan\theta = \cot\phi$$

and therefore

$$m = \frac{1}{a} \qquad (4)$$

Substituting equation **(4)** into equation **(3)** makes the first term in equation **(3)** vanish, and the remaining equation is

$$\frac{y^2}{a^2} - \frac{2}{a}bx - b^2 = 0.$$

Multiply both sides by a^2**:**

$$y^2 - 2abx - a^2b^2 = 0$$

Now solve for x**:**

$$x = \frac{1}{2ab}y^2 - \frac{ab}{2}. \qquad (5)$$

Equation **(5)** describes a parabola whose axis is parallel to the $x-$axis. Other versions of equation **(5)** can be obtained by rotating the plane around the $z-$axis.

Equation of the ellipse

An ellipse arises when the sum of the angles θ and ϕ is less than a right angle:

$$\theta + \phi < \frac{\pi}{2} \qquad \text{(ellipse)}$$

This means that the tangent of the sum of these two angles is positive:

$$\tan(\theta + \phi) > 0.$$

Using the trigonometric identity

$$\tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$$

we see that the condition on the two angles may be written as follows:

$$\tan(\theta + \phi) = \frac{m + a}{1 - ma} > 0 \qquad (6)$$

But $m + a$ is positive, since the summands are given to be positive, so inequality **(6)** is positive if the denominator is also positive:

$$1 - ma > 0. \qquad (7)$$

From inequality **(7)** we can deduce

$$ma < 1, \quad m^2a^2 < 1, \quad 1 - m^2a^2 > 0,$$

$$\frac{1}{m^2a^2} > 1, \quad \frac{1}{m^2a^2} - 1 > 0, \quad \frac{1}{a^2} - m^2 > 0 \qquad \text{(ellipse)}.$$

Let us start out again from equation **(3):**

$$x^2\left(\frac{1}{a^2} - m^2\right) + \frac{y^2}{a^2} - 2mbx - b^2 = 0, \qquad (3)$$

This time the coefficient of the x^2 term does not vanish but is instead positive. Solving for y we obtain the following:

$$y = a\sqrt{b^2 + 2mbx - x^2\left(\frac{1}{a^2} - m^2\right)} \qquad (8)$$

This would clearly describe an ellipse were it not for the second term under the radical, the $2mbx$. It would be the equation of a circle which has been stretched proportionally along the directions of the $x-$ axis and the $y-$ axis. Equation **(8)** is an ellipse but it is not obvious, so it will be rearranged further until this is obvious. Complete the square under the radical, so that the equation transforms into

$$y = a\sqrt{b^2 - \left[x\sqrt{\frac{1}{a^2} - m^2} - \frac{b}{\sqrt{\frac{1}{a^2m^2} - 1}}\right]^2 + \left(\frac{b^2}{\frac{1}{a^2m^2} - 1}\right)}.$$

Now group together the b^2 terms:

$$y = a\sqrt{b^2\left(1 + \frac{1}{\frac{1}{a^2m^2} - 1}\right) - \left[x\sqrt{\frac{1}{a^2} - m^2} - \frac{b}{\sqrt{\frac{1}{a^2m^2} - 1}}\right]^2}.$$

Next, divide by a then square both sides:

$$\frac{y^2}{a^2} + \left(x\sqrt{\frac{1}{a^2} - m^2} - \frac{b}{\sqrt{\frac{1}{a^2m^2} - 1}}\right)^2 = b^2\left(1 + \frac{1}{\frac{1}{a^2m^2} - 1}\right)$$

The x term in the preceding expression has a mildly complicated coefficient, and it will be useful to pull it out by factoring it out of the second term, which is a square:

$$\frac{y^2}{a^2} + \left(\frac{1}{a^2} - m^2\right)\left(x - \frac{b}{\sqrt{\left(\frac{1}{a^2m^2} - 1\right)\left(\frac{1}{a^2} - m^2\right)}}\right)^2 = b^2\left(1 + \frac{1}{\frac{1}{a^2m^2} - 1}\right)$$

Further rearrangement of constants finally leads to

$$\frac{y^2}{1-a^2m^2} + \left(x - \frac{mb}{\frac{1}{a^2} - m^2}\right)^2 = \frac{a^2b^2}{(1-a^2m^2)^2}.$$

The coefficient of the *y* term is positive (for an ellipse). Renaming of coefficients and constants leads to

$$\frac{y^2}{A} + (x - C)^2 = R^2 \qquad (9)$$

which is clearly the equation of an ellipse. That is, equation **(9)** describes a circle of radius R and center $(C, 0)$ which is then stretched vertically by a factor of sqrt (A). The second term on the left side (the $x-$ term) has no coefficient but is a square, so that it must be positive. The radius is a product of squares, so it must also be positive. The first term on the left side (the $y-$ term) has a coefficient which is positive, and hence the equation describes an ellipse.

Equation of the hyperbola

The hyperbola arises when the angles θ and ϕ add up to an obtuse angle, which is greater than a right angle. The tangent of an obtuse angle is negative. All the inequalities which were valid for the ellipse become reversed. Therefore we have

$$1 - a^2m^2 < 0 \qquad \text{(hyperbola)}.$$

Otherwise the equation for the hyperbola is the same as equation **(9)** for the ellipse, except that the coefficient A of the y term is negative.

Outline of Apollonius' books ***<u>On Conics</u>***

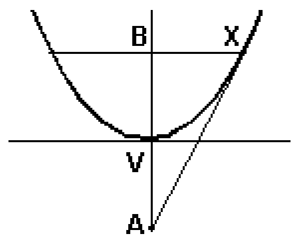
We have already discussed some general features of Apollonius's influential writings on conics, and we shall now summarize the contents of this work a little more specifically. The first four books give a systematic account of the main results on conics that were known to earlier mathematicians such as Manaechmus, Euclid and Aristaeus the Elder, with several improvements due to Apollonius himself. This is particularly true for Books **III** and **IV**; in fact, the majority of results in the latter were apparently new.

One distinguishing property of noncircular conics is that they determine a pair of mutually perpendicular lines that are called the ***major*** and ***minor axes***. For example, in an ellipse the major axis marks the direction in which the curve has the greatest width, and the minor axis marks the direction in which the curve has the least width. Apollonius analyzes these axes extensively throughout his work. Here are a few basic points covered in his first four books.

1. Tangents to conics are defined, but not systematically as in analytic geometry and calculus. Instead, tangents were viewed as lines that met the conic (or branch of the conic for hyperbolas) in one point such that all other points of the conic or its branch lie on the same side of that line.
2. Asymptotes to hyperbolas were defined and studied.

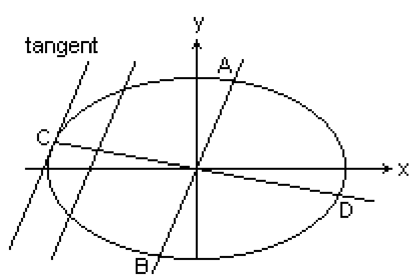
3. Conics were described in terms of (Greek versions of) algebraic second degree equations involving the lengths of certain line segments. Several different characterizations of this sort were given. In many cases these results are forerunners of the algebraic equations that are now employed to describe conics.
4. The intersection of two conics was shown to consist of at most four points.

Here is a typical result from the early books: *Suppose we are given a parabola and a point* **X** *on that parabola that is not the vertex* **V**. *Let* **B** *be the foot of the perpendicular from* **X** *to the parabola's axis of symmetry, and let* **A** *be the point where the tangent line at* **X** *meets the axis of symmetry. Then the distances* **|AV|** *and* **|BV|** *are equal.*



A proof of this result using modern methods is given in an addendum (**4.A**) to this unit; in principle, this result is equivalent to saying that the derivative of x^2 is $2x$.

Books $\mathbf{V} - \mathbf{VII}$ of ***On Conics*** are highly original. In Book $\mathbf{V}$, Apollonius considers normal lines a conic; these are lines containing a point on the conic that are perpendicular to the tangents at the point of contact. As in calculus, Apollonius' study of such perpendiculars uses the fact that they give the shortest distances from an external point to the curve. Book $\mathbf{V}$ also discusses how many normal (or perpendicular) lines can be drawn from particular points, finds their intersections with the conics by construction, and studies the curvature properties in remarkable depth. In particular, at each point there is a center of curvature, which yields the best circular approximation to the conic at that point, and Apollonius' results on finding the center of curvature resolve this issue completely. A main objective of Book $\mathbf{VI}$ is to show that the three basic types of conics are geometrically dissimilar in roughly the same way that, say, a triangle and a rectangle are dissimilar. In Book $\mathbf{VII}$, Apollonius deals with the various relationships between the lengths of diameters and their ***conjugate diameters***, which are defined as follows: Given a diameter **AB** of the conic (which passes through the center of the conic), the endpoints of the conjugate diameter **CD** are points such that the tangents to the conic at **C** and **D** are parallel to **AB** (see the drawing below).



Conjugate Diameters

(Source: http://mysite.du.edu/~jcalvert/math/ellipse.htm)

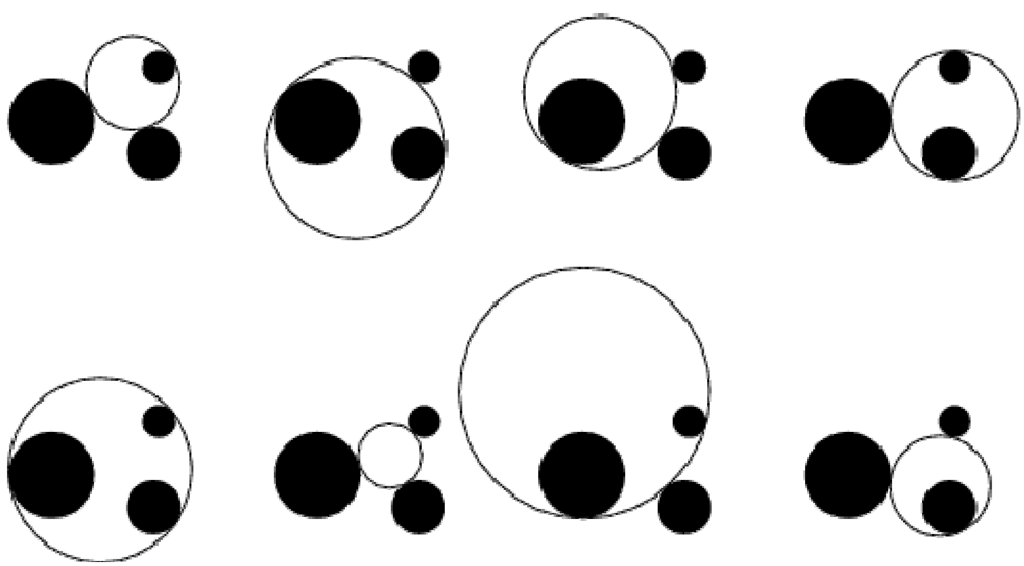
The results are applied to the exposition of a number of problems, as well as to some problems which Apollonius indicates will be demonstrated and solved in Book VIII, which was lost in Antiquity. The final portion of the work contains (or is reputed to contain) further results involving major and minor axes and their intersections with the conics.

A treatment of Apollonius' work in (relatively) modern terms is given in the following book:

> H. G. Zeuthen, ***Die Lehre von den Kegelschnitten im Altertum*** (*The study of the conic sections in antiquity*; translation from Danish into German by R. von Fischer-Benzon), A. F. Höst & Son, Copenhagen, DK, 1886. See the file **http://math.ucr.edu/~res/math138A-2012/zeuthen.pdf** for an online copy from Google Book Search.

The Problem of Apollonius

In an essay on ***Tangencies***, Apollonius is also known for posing the following general problem (frequently called the ***Problem of Apollonius***): *Given three geometric figures, each of which may be a point, straight line, or circle, construct a circle tangent to the three; the most difficult case arises when the three given figures are circles.* — Trial and error frequently yields explicit solutions to this problem in specific instances, and in fact one can see that there are up to **8** different solutions in some cases.



(Source: http://mathworld.wolfram.com/ApolloniusProblem.html)

Apollonius claimed to have solved this problem, but his solution is lost. There is also further information on this topic in the following online sites:

http://www.ajur.uni.edu/v3n1/Gisch%20and%20Ribando.pdf

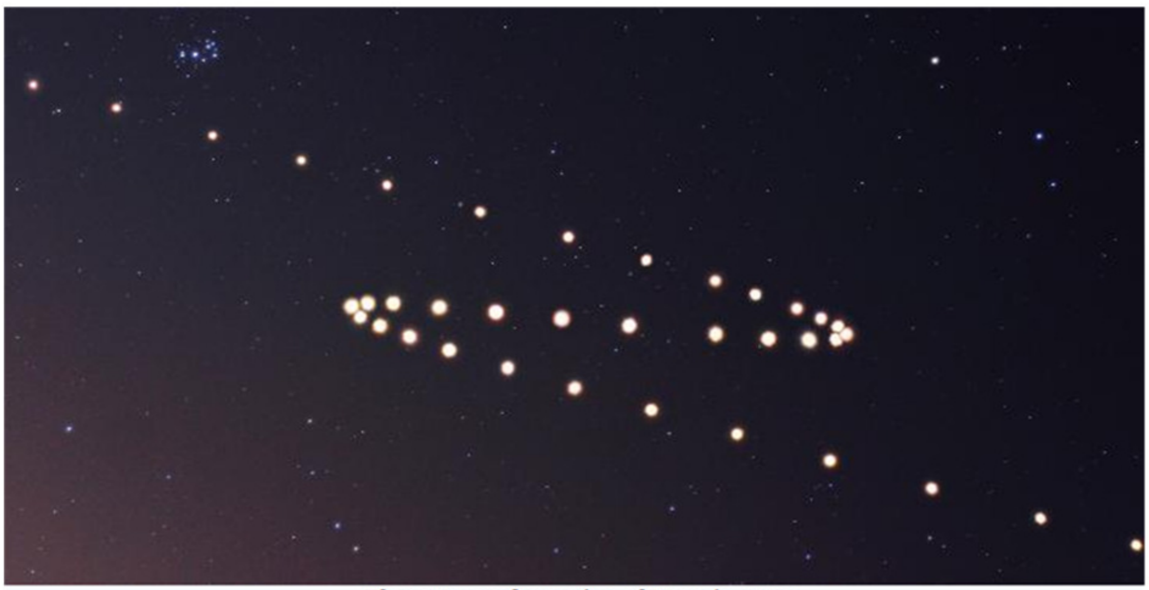
http://en.wikipedia.org/wiki/Problem_of_Apollonius

http://jwilson.coe.uga.edu/emt725/Apollonius/Prob.Apol.html

Other works of Apollonius

Most of Apollonius' other works are lost, but we have some information about this work from the writings of others. In a computational work called ***Quick Delivery*** he gave an estimate of 3.1416 for π that was better than the more commonly used Archimedean estimate of $22/7$. We shall only mention two other items on the list.

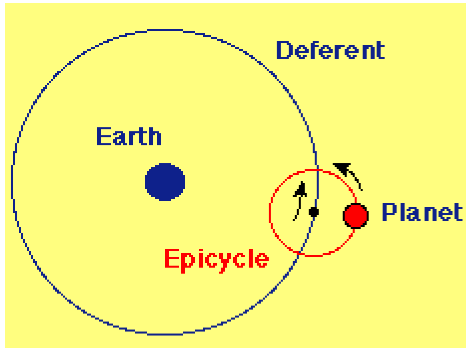
The first involves Apollonius' work on mathematical astronomy. His view of the solar system was that the sun rotated around the earth but the remaining planets rotated around the sun. This clearly differs from the more widely held belief that everything rotated about the earth. However, well before his time astronomical observations showed beyond all doubt that the planets did not move around the earth in perfectly circular orbits. If this were the case, then just like the moon the planets' observed paths across the sky would be straight from east to west, but astronomical observations show that sometimes the planets seem to move backwards (***retrograde*** motion). The following pictures for the motion of Mars illustrate this phenomenon:



The retrograde motion of Mars in 2005.
A composite image created by superimposing images taken on 35 different dates, each separated from the next by about a week. (Tunc Tezel, apod060422)

(Source: http://cseligman.com/text/sky/retrograde.htm)

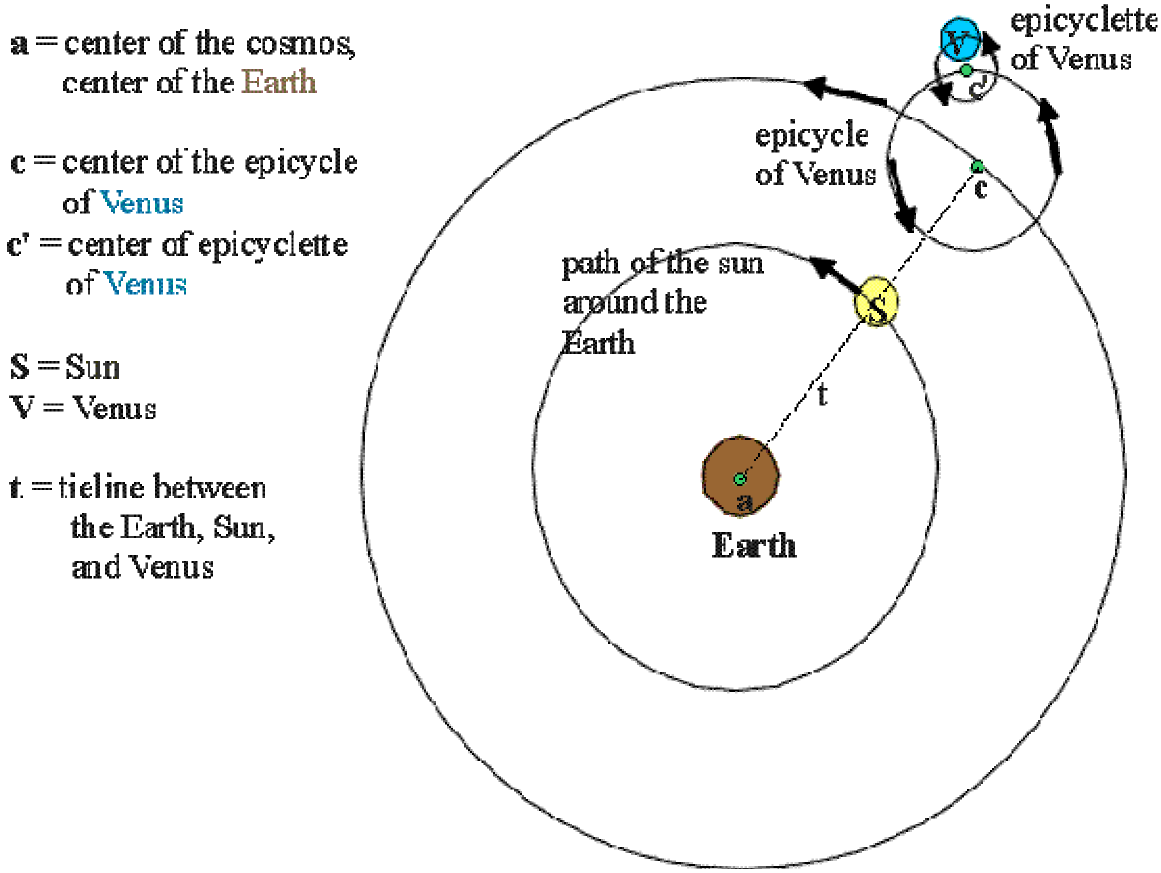
Apollonius explanation for planetary motion evolved indirectly into a cornerstone of Claudius Ptolemy's later work in the 2^{nd} century A.D.. A major feature of Ptolemy's theory was a hypothesis that the planets moved in combinations of circles which are called ***epicycles***. The idea is similar to our concept of the Moon's motion around the earth; namely the moon moves around the earth in an ellipse while the earth in turn moves around the sun in another ellipse. However, in Apollonius' (and Ptolemy's) setting the curves were circles rather than ellipses and there was no actual mass corresponding to the center of the smaller circle. Here is a simple illustration of epicycles:



(Source: http://www.cartage.org.lb/en/themes/Sciences/Astronomy/TheUniverse/Oldastronomy/TheUniverseofAristotle/TheUniverseofAristotle.htm)

There are also some animated graphics on the site from which this picture was taken.

A more elaborate illustration of this motion model is illustrated below; in this example, there is in fact a second epicycle moving around the first one. The Ptolemaic theory of planetary motion required ***dozens*** of such higher order epicycles.



(***Source:*** http://inst.santafe.cc.fl.us/~jbieber/HS/ptol_epi.htm)

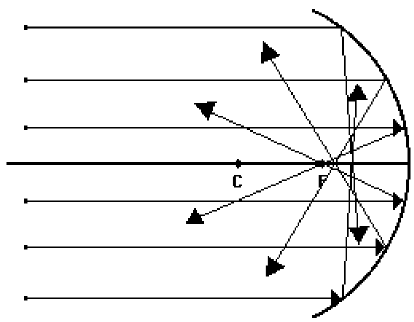
Given Kepler's discovery that planets move in elliptical paths around the sun, it is somewhat ironic that the author of the definitive classical work on conic sections proposed motion by epicycles instead, but that is what happened. Here are a few other links to pictures of epicycles, some with animation:

http://www.opencourse.info/astronomy/introduction/05.motion_planets/

http://www.math.tamu.edu/~dallen/masters/Greek/epicycle.gif

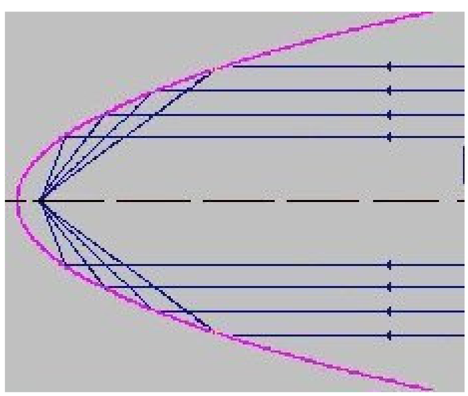
http://www.edumedia.fr/animation-Epicycle-En.html

Apollonius and his contemporary Diocles (240 – 180 B.C.E.) are also given credit for discovering the ***reflection property of the parabola***. The Greeks knew that one could start a fire by focusing the sun's rays using a convex mirror; stories that Archimedes used large mirrors of this sort to set fire to Roman ships are almost certainly incorrect, but the idea was known at the time. The simplest concave mirrors are shaped like a portion of a sphere. However, these do not have a true focus but suffer from a phenomenon called *spherical aberration*.



(*Source:* http://www.glenbrook.k12.il.us/gbssci/phys/Class/refln/u13l3g.html)

The failure of the reflected rays to go through a single point means that a spherical mirror is somewhat inefficient in focusing the sun's rays (or any other rays for that matter), but if one uses a ***parabolic*** mirror this problem is eliminated. All incoming light rays parallel to the axis of symmetry will then be reflected to the focus of the parabola. This property of the parabola is used extensively for devices like antennas and radio telescopes that are designed to receive and focus electromagnetic waves.

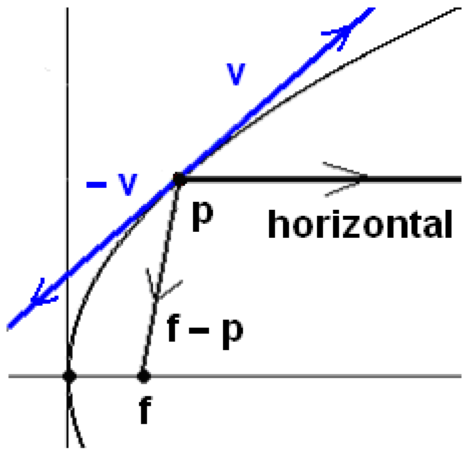


(Source: http://library.thinkquest.org/23805/math1.htm)

Proving the reflection property of a parabola is basically an exercise in geometry, and the following online site contains a proof using methods from elementary geometry:

http://www.pen.k12.va.us/Div/Winchester/jhhs/math/lessons/calculus/parabref.html

Needless to say, one can also derive the reflection property for a parabola using vector and/or coordinate geometry. Here is a sketch of the proof: First, choose coordinates and measurement units so that the equation of the parabola is given by $y^2 = 4ax$. Then the focus **f** of the parabola turns out to have coordinates $(a, 0)$. Next, let **p** = $(b^2/4a, b)$ be a point on the parabola, and for the sake of convenience suppose that b is positive (note that the curve is symmetric with respect to the x – axis). Then the direction of the tangent vector at **p** is given by **v** = $(2b, 4a)$, and proving the reflection property amounts to showing that the angle between **v** and the horizontal unit vector $(1, 0)$ is equal to the angle between **–v** and **f** – **p**, which is the same as the angle between **v** and **p** – **f** (see the figure below).



(**Source: http://www.analyzemath.com/parabola/parabola_work.html**)

It is enough to show that the cosines of the two angles between the pairs of vectors are the same, and since the vectors **p**, **f** and **v** are given explicitly in terms of a and b this is essentially an exercise in using the standard dot product formula for the cosine of the angle between two vectors.

Ellipses also have an important reflection property, and it is discussed in the following online document:

http://usiweb.usi.edu/students/gradstudents/j_k_l/kleinknecht_s/portfolio/Educ%20690_004%20ST/History%20of%20Conics.htm

As noted in that reference, one physical consequence of the reflection property is the "whispering gallery" phenomenon; if we are given a room shaped like the inside of an elliptical region, then a whispered message at one focus of the ellipse can be heard more clearly at the second focus than at many other points which are closer to the first focus (one example is the Statuary Hall in the U. S. Capitol).

Here is a reference for a mathematical derivation of the reflection property for ellipses:

http://math.ucr.edu/~res/math153/ellipse-reflection.pdf