5.C. Geometric approaches to Diophantine equations

Our purpose here is to analyze a problem from Book IV of Diophantus' work, *Arithmetica*. As is often the case, the problem was originally stated more specifically with some explicit numbers.

THEOREM. *Let $a > 2$ be a positive rational number. Then there are positive rational numbers x and y such that $y(a-y) \;=\; x^3 - x$.*

Proof. Let Γ be the set of all points (x, y) in the coordinate plane such that x and y are rational and $y(a-y) \;=\; x^3 - x$. Then $(-1, 0) \in \Gamma$, and we shall find a point with the desired properties by considering the intersections of Γ with lines through $(-1, 0)$.

More precisely, we shall consider lines with equations of the form $x = ty - 1$ for some rational value of t. Each of these lines meets Γ at $(-1, 0)$, and for some choice of t we want to find a second point in this intersection such that both coordinates x, y are positive rational numbers with $0 < y < a$. One condition for a point (x, y) to lie on Γ and the line is

$$y(a-y) \;=\; (ty-1)^3 \;-\; (ty-1) \;=\; t^3y^3 \;-\; 3t^2y^2 \;+\; 2ty\ .$$

If we divide both sides of this equation by y, we obtain a quadratic equation in y, and if we fix t then we can solve this to find the y-coordinates for all points on the curve. We need to find a value of t for such that x and y are positive rational numbers with $0 < y < a$.

The equation for y can be rewritten in the form

$$0 \;=\; t^3y^3 \;-\; (3t^2-1)y^2 \;+\; (2t-a)y$$

and if we choose $t = a/2$ so that the first degree terms vanishes, then we are left with the equation

$$0 \;=\; \frac{a^3}{8}\,y^3 \;-\; \left(\frac{3a^2}{4} - 1\right)\,y^2$$

and since a is rational it follows that all roots of this equation are also rational.

Clearly the unique nonzero solution to the preceding equation is

$$y \;=\; \frac{2(3a^2-4)}{a^3} \;=\; \frac{6a^2-8}{a^3}\ .$$

which is positive because its numerator is positive when $a^2 > \frac{4}{3}$ and we know that $a^2 > 4$. To prove that $y < a$, note that this is translates to

$$\frac{6a^2-8}{a^3} \;<\; a \quad \text{or equivalently} \quad 6a^2 - 8 < a^4$$

and $a^4 - 6a^2 + 8$ is positive if $a^2 > 4$; since a is positive this is equivalent to $a > 2$.

It follows that $x = ty - 1$ is given by

$$\frac{a}{2}\left(\frac{6a^2-8}{a^3}\right) \;-\; 1 \;=\; \frac{3a^2-4}{a^2} \;-\; 1$$

so that $x > 0$ (what we want) if and only if $3 - 4a^{-2} > 1$. The latter inequality is equivalent to $a^2 > 2$, and since we are assuming that $a > 2$ we can also conclude that $x > 0$.■

Special case. If we choose $a = 6$ as Diophantus does, then we obtain the solution

$$(x, y) \quad = \quad \left(\frac{136}{27}, \frac{26}{27} \right) .$$

Integer solutions

One can also ask if the equation has integral solutions, and if we take Diophantus' choice of $a = 6$ the answer is affirmative. In fact, one has the following solutions in this case, but note that in each example either x or y is negative:

$$(x, y) \quad = \quad (-9, 30) \ , \quad (-9, -24) \ , \quad (-35, 210) \ , \quad (-34, -204) \ , \quad (-37, 228) \ , \quad (-37, -222)$$

The following book discusses of this and other problems in Diophantus' *Arithmetica* at the undergraduate level:

I. G. Basmakova, *Diophantus and Diophantine Equations* (Transl. by A. Shenitzer, with an Addendum by J. H. Silverman), Mathematical Association of America Dolciani Expositions No. 20. Mathematical Association of America, Washington, DC, 1997.