6. Mathematics of Asian and Arabic civilizations — II

(Burton, **5.3, 5.5, 6.1**)

Due to the length of this unit, it has been split into two parts.

Arabic/Islamic mathematics

We have already mentioned that the term "Greek mathematics" refers to a fairly wide geographic area and contributions of mathematicians of many nationalities. The same can be said about the mathematics associated to Arabic and Islamic cultures which flourished during the period from about 800 to 1500, but the geographic area, the diversity of nationalities, and even the diversity of religions was much greater than in Greek mathematics. The geographic range included the entire Islamic world at the time, from Spain on the west to Uzbekistan on the east (see **http://math.ucr.edu/~res/historical-maps2.pdf** for typical maps). Furthermore, many of the major figures had non – Arabic ethnic roots (for example, as elsewhere in Islamic culture, Persian contributions were very extensive), and the role of Jewish scholars, largely in Spain, is particularly apparent (similar levels of participation and acceptance did not occur in mainstream European mathematics until the 19th century). When using phrases like "Arabic mathematics" or "Islamic mathematics" it is important to remember the geographic, ethnic and religious diversity of those who worked within this framework. In analogy with Greek mathematics, an important unifying factor is that Arabic was the language for most of the written output. Some Iranian bloggers have objected to terms like "Arabic mathematics" or "Arabic science" because these do not indicate the extremely important role of Persian scholars, and such complaints are perfectly understandable. Unfortunately, there is no simple way to describe this body of scientists that recognizes both its ethnic and religious diversity but fits easily into less than half of a single printed line, and in these notes we have chosen a frequently employed "abuse of language" (with the preceding stipulation!) for the sake of conciseness.

Since the center of the Islamic world was, and still is, between Europe and the Indian subcontinent, it is not surprising that Arabic mathematics was heavily influenced by both Greek and Indian mathematics. For our purposes, two absolutely crucial legacies of Arabic mathematics are the following (see the final paragraph of **(6.A)** for more on the Arabic legacy):

(1) It preserved a very substantial amount of classical Greek mathematics that would otherwise have been lost or ignored.

(2) It also passed along the important new insights that Indian mathematicians had discovered.

In addition to these contributions, numerous Arabic mathematicians also made several important and highly original contributions, a great many of which were independently rediscovered by (non – Arabic) European mathematicians, in some cases several centuries later. Unfortunately, for reasons of space we shall only be able to give a few examples in these notes.

The emergence of algebra

> The "Arabic miracle" lies not so much in the rapidity with which the political empire rose as in the alacrity with which, their tastes aroused, the Arabs absorbed the learning of their neighbors.
>
> Boyer & Merzbach, ***A History of Mathematics*** (2nd Ed.), p. 227

One of the best-known names in Arabic/Islamic mathematics was also one of the earliest: Abū 'Abdallāh Muḥammad ibn Mūsā **al-Khwārizmī** (790 – 850). He is known for two major pieces of work. The first was an extremely influential account of the Hindu numeration system based upon an earlier Hindu work. Although the original Arabic versions of this work are lost, significantly altered Latin translations have survived, and typical Latin transliterations of his name as ***Algoritmi*** or ***Algorism*** have evolved into our modern word ***algorithm***. A second piece of his work has given us another basic word in mathematics. The word ***algebra*** comes directly from his book, *Hisab* ***al-jabr*** *w'al-muqabala* (often translated with a phrase like "The Science of Restoration and Reduction"). An extended discussion of this work's contents appears on pages 238 – 244 of Burton, so we shall concentrate here on the nature of Al-Khwarizmi's contributions. There are significant differences of opinion about this. The problem solving methods in the work can be found in earlier writings of others, and the notation does not contain any significant advances. For example, everything is done using words, and there is no shorthand notation analogous to that of Diophantus. In contrast to the work of Greek mathematicians (including Diophantus), rational and irrational numbers are treated the same, just as in Indian mathematics. On the other hand, ***al-jabr*** systematically avoids using negative numbers, unlike the earlier work of Brahmagupta where such numbers were used freely. Possibly the most important innovation in ***al-jabr*** is a major change in viewpoint from the earlier work of the Babylonians and Diophantus. Specifically, ***al-jabr*** takes a highly systematic approach to solving various sorts of equations (especially quadratic equations). The following quotation from page 111 of Hodgkin describeses this new perspective very succinctly:

> Al-Khwarizmi does not wish, like the Babylonians, to list particular cases and assume that you can deduce the general rule; he wants his statements to be general.

There is also a strong emphasis on ***equation solving for its own sake*** rather than the theory of numbers or some other area of interest. Even though there is no symbolism or shorthand in al-Khwarizmi's writings along the lines of Diophantus, equations are frequently discussed using general terminology and words like ***root*** that have become standard mathematical vocabulary. Such discussions resemble modern verbal descriptions of problems, and from this perspective mathematical language appears to follow the pattern of most languages, with verbal formulations of concepts coming before an efficient symbolism is created for writing them down (another example mentioned earlier is the verbal discussion of zero in Brahmagupta's writings centuries before the first known symbolism for it). ***Al-jabr*** makes important progress towards removing extraneous geometrical ideas from solving equations (***e.g.***, as found in Book **II** of Euclid's ***Elements***), and as such plays an important role in separating these subjects from each other. However, the subjects are not completely uncoupled from each other, and in many instances Al-Khwarizmi uses geometrical ideas to prove that his algebraic solution techniques yield correct answers.

Other studies along the lines of ***al-jabr*** took place during or shortly after Al-Khwarizmi's time. One of the most prominent and influential contributors was Abū Kāmil Shujā' ibn Aslam ibn

Muḥammad ibn Shujā (**Abu Kamil,** $c.$ 850 – $c.$ 930), whose work is discussed at some length on pages 242 – 244 of Burton. Among other things, his work drastically reduced the reliance on Greek geometrical methods.

We have already mentioned Thabit ibn Qurra (Al-Ṣābi' Thābit ibn Qurra al-Ḥarrānī, 836 –901) in our discussion of amicable pairs. His criterion for finding such pairs was completely original and was not improved upon until the 18th century. He also translated many important Greek manuscripts into Arabic and made numerous further contributions to mathematics and other subjects; some further discussion of his mathematical work appears on pages 244 – 246 of Burton.

Al-Khwarizmi's separation of algebra from geometry was put into a definitive form by Abū Bakr ibn Muḥammad ibn al Ḥusayn **al-Karajī** (or **al-Karkhī**, 953 – 1029), who also took important steps in defining nonzero integral powers (including negative ones!) algebraically and came very close to discovering the law of exponents $x^n x^m = x^{m+n}$ where m and n are integers; the only missing element was that he did not set x^0 equal to 1. Al-Karaji also used a partially developed form of mathematical induction in some of his proofs, and in particular in an argument essentially showing that

$$1^3 + \ldots + n^3 = (1 + \ldots + n)^2.$$

He also had many key insights into the formalism of polynomials, including powers of binomias and the binomial coefficient identity known as ***Pascal's triangle***, which was also known to the Chinese mathematican Jia Xian ($c.$ 1010 – $c.$1070) and also appears in a 10th century Indian commentary on the ancient work of Pingala:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

To illustrate the extent of some of Al-Karaji's results, we shall discuss his study of the Diophantine equation $x^3 + y^3 = z^2$. This equation has several obvious integral solutions like $(x, y, z) = (1, 2, 3)$ and $(2, 2, 4)$, and Al-Karaji discovered the following parametrization formulas, which show that there are many different ordered triples of positive rational numbers which solve this equation:

$$x = \frac{u^2}{1 + v^3}, \quad y = \frac{u^2 v}{1 + v^3}, \quad z = \frac{u^3}{1 + v^3}$$

There is a striking contrast between this result and a famous theorem of P. de Fermat which states that _the equation_ $x^4 + y^4 = z^2$ _has no nontrivial positive integral solutions_; in fact, this equation also has no rational solutions such that x and y are both positive, for (as usual) if there were such a solution then one could multiply both sides by products of denominators to obtain a similar solution such that x and y are both positive integers. Here is a brief but clearly written proof of Fermat's result:

http://www.cem.uvm.edu/~cooke/history/seconded/outtakes/sec81.pdf

Al-Karaji's work on polynomials and mathematical induction was carried significantly further by Ibn Yaḥyā al-Maghribī **al-Samaw'al** (also known as Samau'al al-Maghribi, $c.$ 1130 – $c.$ 1180). In particular, his results include general forms of standard modern techniques for working with polynomials, most notably including the long division principles for polynomials in elementary algebra (see **http://www.purplemath.com/modules/polydiv3.htm** for a standard example**).** A few specific results of his are discussed at some length on pages 117 – 120 of Hodgkin's book**.**

More generally, there is also a fairly extensive analysis of Arabic mathematicians' contributions to algebra and the Hindu – Arabic numeration system (including early studies of decimal – like expansions) in Chapter 5 of Hodgkin**.**

Arabic work on trigonometry

Arabic mathematics is particularly known for its extensive contributions to the development of trigonometry**,** including extensive work on computing values of trigonometric functions to ever increasing degrees of accuracy**;** the underlying motivation was the importance of trigonometry for astronomical measurements**,** but their discoveries of trigonometric identities were of far – reaching importance**.**

One early and highly influential contributor to the subject was **'**Abū **'**Abd Allāh Muḥammad ibn Jābir ibn Sinān ar-Raqqī al-Ḥarrānī aṣ-Ṣābi**' al-Battānī** (Latinized as Albategnius**,** Albategni or Albatenius**,** 850 – 929)**.** His astronomical tables had a very strong impact upon European astronomy into the 16th century**,** and he discovered a number of basic trigonometric identities, particularly involving the tangent function which he studied extensively**.**

Another important figure from the 10th century was Abū al-Wafā' Muḥammad ibn Muḥammad ibn Yaḥyā ibn Ismā'īl ibn al-'Abbās al-Būzjānī (**Abul Wafa Buzjani,** 940 – 998)**.** Among other things**,** he worked extensively with all six of the basic trigonometric functions**.** He also devised new methods for computing sines of angles**,** and compiled tables of values with incremental intervals of $\mathbf{0.25}$ degrees**;** in modern decimal notation his results were accurate to $\mathbf{8}$ decimal places**.** By contrast**,** the tables of Claudius Ptolemy were accurate to three decimal places in modern notation (at the time values were expressed in Babylonian sexagesimal form**,** a practice which continued for a few centuries longer)**.** In another direction**,** he discovered the Law of Sines for spherical triangles (see **http://mathworld.wolfram.com/SphericalTrigonometry.html** for a discussion of spherical trigonometry) and the formula for the sine of a sum of two angles**.** Abul Wafa is also known for his writings on geometry**;** the latter discuss at length the repeating**,** abstract geometric patterns that play an important role in Islamic art and architecture**.** The following online reference summarizes these geometrical writings**:**

http://www.mi.sanu.ac.rs/vismath/sarhangi/index.html

The following reference describes a recent discovery of a relation between Islamic designs and geometric questions which mathematicians have studied during the past half century**:**

http://www.sciencenews.org/view/generic/id/8270/title/Math_Trek__Ancient_Islamic_Penrose_Tiles

For background information see **http://en.wikipedia.org/wiki/Penrose_tiling.**

Although Abu al-Hasan 'Ali abi Sa'id 'Abd al-Rahman ibn Ahmad **ibn Yunus** al-Sadafi al-Misri (**Ibn Yunus,** 950 – 1009) is best known for extensive and highly reliable astronomical observations**,** he is also credited with discovering the key trigonometric identity

$$\cos s \;+\; \cos t \;=\; 2 \cos \tfrac{1}{2}(s+t) \cos \tfrac{1}{2}(s-t)$$

which led to the Law of Cosines in spherical trigonometry and was often used to transform complicated multiplicative expressions into additive expressions, much in the same way that logarithms have been used since the 17th century.

Possibly the first mathematician to separate trigonometry from astronomy was Muḥammad ibn Muḥammad ibn Ḥasan Ṭūsī (better known as **Naṣīr al-Dīn al-Ṭūsī** or **Nasireddin,** 1201 – 1274). His work put spherical trigonometry into a comprehensive definitive form, and his treatment also included the Law of Sines for plane triangles. He is also known for studying a logical alternative to Euclid's Fifth Postulate and for a construction called the ***Tusi couple***, which expresses linear motion as the result of two rotational motions and was subsequently used by others, possibly including N. Copernicus (1473 – 1543), in refining or developing theories to explain planetary motion. Here is an animated reference for the Tusi couple:

http://mathworld.wolfram.com/TusiCouple.html

There was also an earlier mathematician with the similar name of Sharaf al-Dīn al-Muẓaffar ibn Muḥammad ibn al-Muẓaffar al-Ṭūsī (**Sharaf al-Tusi,** 1135 – 1213); he is known for his work on finding solutions to cubic equations by geometrical methods, which included special cases of some basic concepts from differential calculus.

Other major contributors

We have already noted that the number of contributors to Arabic mathematics was quite large, and in fact there are too many names to include in a set of notes at this level, so we shall only discuss a few individuals who were particularly well – known.

One of the most important figures in Arabic science during this period was Abū ʿAlī al-Ḥasan ibn al-Ḥasan ibn **al-Haytham** (965 – 1039), who is frequently called ***Alhazen*** or something similar in European languages. His experimental observations revolutionized the subject of optics, and this work forcefully demonstrated the importance of carrying out experiments in scientific studies; his book, *Kitab al-Manazir* (*Book of Optics*), has been viewed as one of the most important contributions to physics because of its results and its description of an early version of the scientific method. Greek theories of optics postulated that we see objects because of rays emitted by our eyes or because of physical objects entering our eyes, but Alhazen showed we see objects because of rays coming from the objects themselves. His mathematical work involved a number of problems, several of which anticipated much later discoveries, but perhaps his best known mathematical contribution is his work on the ***Alhazen billiard problem*** (actually first considered by Claudius Ptolemy), which is to find a point on a spherical mirror at which light from a source will be reflected to the eye of an observer. Alhazen shows that the problem can be solved fairly directly using conic sections; here is one reference for such a solution:

http://www.ibguide.org/files/Download/ZouevAlexanderExtendedEssayFinal.pdf

However, it turns out that the Alhazen billiard problem cannot be solved by straightedge and compass, and this was first shown in 1997 by P. M. Neumann (1940 –). Here is the reference:

> **P. M. Neumann.** ***Reflections on reflection in a spherical mirror.*** American Mathematical Monthly **105** (1998), 523 – 528.

Normally we associate the name **Omar Khayyám** (Ghiyās od-Dīn Abol-Fath Omār ibn Ebrāhīm Khayyām Neyshābūri, 1048 – 1122) with the poetic work *Rubaiyat*, and its 19th century English translation by E. FitzGerald (1809 – 1883), but Khayyám also made significant contributions to other areas, and mathematics is a notable example. We shall only mention one example here. Recall that the Greek mathematician Menaechmus solved the problem of duplicating a cube by using two intersecting parabolas to construct a segment whose length was **cbrt(2).** Khayyám gave methods for geometrically finding roots of more general cubic equations by constructing various conic sections and taking their intersection points. For example, in one type of problem the root is given by intersecting a circle and a parabola. This case is discussed in detail on pages 247 – 249 of Burton; two others are presented in Exercise 7 on page 263 of that text. His approach contrasts sharply with most Arabic works on solving cubic equations, which were aimed at finding highly accurate numerical approximations to roots; Khayyám's geometric approach was a systematic investigation of exact solutions for such equations.

Another important contributor to the study of conic sections was **al-Kūhī** (Abū Sahl Wayjan ibn Rustam al-Qūhī, *c.* 940 – *c.* 1000), who also contributed to astronomy and is known for describing a generalized version of a compass for constructing conic curves more general than circles. Much of his mathematical work dealt with problems resembling those studied by Archimedes and Apollonius, and the following reference summarizes some particularly noteworthy examples:

(*Source:* http://jwilson.coe.uga.edu/EMT668/EMT668.Folders.F97/Norton/Al-Quhi/Tangent_Circles.html)

One of the last major figures in Arabic/Islamic mathematics was Ghiyāth al-Dīn Jamshīd ibn Mas'ūd **al-Kāshī** (or Ghiyāth al-Dīn Jamshīd Kāshānī, 1380 – 1429). There is a brief discussion of his work on pages 250 – 251 of Burton. Al-Kashi is also known for other achievements beyond those cited in Burton, including a definitive form of the Law of Cosines and extensive, systematic use of decimal fractions at a level comparable to (and in some respects better than) the form described by S. Stevin (1548 – 1620) over a century later. As noted on page 120 of Hodgkin's book, there is evidence that al-Kashi's work influenced 16th century European mathematics, but the precise extent of this impact is undetermined. There is also some difference of opinion about whether decimal expansions as we know them are contained in much earlier work by Abu'l Hasan Ahmad ibn Ibrahim **al-Uqlīdisī** (literally "the Euclidian," *c.* 920 – *c.* 980) and/or al-Samawal; Hodgkin's book presents one perspective on this, and the MacTutor biographical article on al-Uqlidisi summarizes the various opinions, noting that "There is no disagreement … that al-Uqlidisi made a major step forward." The extent of al-Samawal's contribution is difficult to assess at this point because the source is a manuscript that was discovered relatively recently and has not yet been translated from Arabic (see Hodgkin, p. 120).

The last mathematician from the Arabic/Islamic school to appear in either the MacTutor site or ***Wikipedia***'s mathematical biographies is Abū al-Hasan ibn Alī **al-Qalasādī** (1412 – 1482), whose contributions to developing mathematical notation will be mentioned in a subsequent unit. It seems appropriate to conclude with Hodgkin's remark that "this may have been the 'end' of mathematics so far as our classical histories go, but the [underlying intellectual and cultural] society [wa]s far from being in decline." Scholarship continued at a high level, but it did so in different directions.