7. Mathematical revival in Western Europe

(Burton, **6.2 – 6.4, 7.1**)

Although mathematical studies and discoveries during the Dark Ages in Europe were extremely limited, there were contributors to the subject during the period from the Latin commentator Boëthius (Anicius Manlius Severinus Boëthius, 475 – 524) shortly after the fall of the Western Roman Empire and continuing through to the end of the 12th century; Boëthius has been described as the last of the Romans and the first of the scholastic philosophers (predating the latter by many centuries), and his mathematical texts of were still widely used 600 years after they were written. Several other names from this period are mentioned in Sections 5.4 and 6.1 of Burton, and the latter's exercises also mention Alcuin of York (735 – 804) and Gerbert d'Aurillac (940 – 1003), who later became Pope Sylvester II (997 – 1003).

During the second half of the 11th century some important political developments helped raise European's consciousness of ancient Greek mathematical work and the more recent advances by Indian and Arabic mathematicians. Many of these involved Christian conquests of territory that had been in Muslim hands for long periods of time. Specific examples of particular importance for mathematics were the Norman conquest of Sicily in 1072, the Spanish ***reconquista*** during which extensive and important territories in the Iberian Peninsula changed from Muslim to Christian hands, and the start of the Crusades in 1095. From a mathematical perspective, one important consequence was dramatically increased access to the work of Arabic mathematicians and their translations of ancient Greek manuscripts. Efforts to translate these manuscripts into Latin continued throughout the 12th century; the quality of these translations was uneven for several reasons (for example, in some cases the Arabic manuscripts were themselves imperfect translations from Greek, and in other cases the translations were based upon manuscripts in very poor condition), but this was an important step to promoting mathematical activity in Europe. A more detailed account of this so – called *Century of Translation* appears on pages 272 – 277 of Burton.

Fibonacci

Leonardo of Pisa (Leonardo Pisano Bigollo), more frequently known by the 18th century nickname ***Fibonacci*** (1170 – 1250), symbolizes the revival of mathematical activity in Europe during the late Middle Ages, and his book ***Liber abaci*** (Book of Counting — literally, the abacus), which appeared in 1202, is the first major work aimed specifically at a European audience that recounts some important ideas from Hindu and Arabic mathematics and integrates this work with that of earlier contributions from Greek mathematics. The work is not merely a routine compilation of material from other sources, but rather it represents an independent and broadly based point of view.

Despite the impact of ***Liber abaci*** during the late Middle Ages, the first printed version did not appear until 1857, nearly 650 years after it was first written; incidentally, the first published English translation of this work appeared in 2003.

Comments on the contents of Fibonacci's writings

Certainly the most far – reaching aspect of ***Liber abaci*** is its presentation of the Hindu – Arabic number system and the large amount of evidence it produces to demonstrate the superiority of the Hindu – Arabic notation and Indian methods of computation. However, there are several other noteworthy features. Some ideas in the book were very advanced for that time, but many aspects of the notation are clumsy by modern standards.

As Burton notes, it is somewhat ironic that today Fibonacci is best known for one problem from his book that was named after him in 1877 by E. Lucas (1842 – 1891); indeed, this sequence appears in writings of the Indian mathematicians Hemachandra (1089 – 1173) and Gospala around 1135, and it also appears in even earlier 7th century Indian writings (and perhaps 1000 years earlier in the writings of Pingala).

There is an extensive discussion of the Fibonacci sequence in Burton in Section 6.3 (pages 287 – 293). Perhaps the most notable omission is an explicit formula for the values of F_n as a function of n. The formula is given by

$$F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$$

and a derivation is given in **http://math.ucr.edu/~res/math153/history07a.pdf**. For the sake of completeness, we shall also give references for a few other formulas which are stated and used in the exercises on pages 292 – 293 of Burton. In Exercises 5 and 6, one needs the fact that F_d divides F_n if and only if d divides n. This is relatively simple to prove and can be found on pages 51 – 59 of the following book:

> **N. N. Vorobiev. *Fibonacci Numbers*** (Transl. from the 1992 Russian original by M. Martin). Birkhäuser Basel, Boston MA, 2002.

On the other hand, the divisibility statement for Fibonacci numbers in Exercise 3 is a deeper fact; here is an online reference:

http://math.arizona.edu/~ura-reports/071/Campbell.Charles/Final.pdf

Exercise 1 cites a weak form of remarkable result called ***Zeckendorf's Theorem***; a description and some references are given in the following online references:

http://en.wikipedia.org/wiki/Zeckendorf%27s_theorem

http://mathworld.wolfram.com/ZeckendorfRepresentation.html

The Fibonacci sequence has been studied extensively, with some properties fairly easy to verify while others are quite difficult and involve highly sophisticated methods. There is an extensive summary in **http://mathworld.wolfram.com/FibonacciNumber.html**.

In ***Liber quadratorum*** Fibonacci investigates several ***number – theoretic questions involving perfect squares***. Much of this is described on pages 280 – 281 of Burton. As noted there, one basic topic of interest in this book is the following:

> *Given an arithmetic progression of integers* $a + nd$ (*where* a *and* d *are fixed integers and* n *is the variable*), *can one find three or more successive values that are perfect squares*?

Burton gives one example of such a triple — namely, $\{1, 25, 49\}$ — for which the difference d is equal to 24, and some of the exercises discuss additional aspects of this question. Remarks in Burton suggest that other triples of this type exist, but nothing further is stated, so we shall fill in some details here. In fact, there are 67 sequences of three elements whose constant difference is $10{,}000$ or less. The next few are given as follows:

$$\{2^2, 10^2, 14^2\}, \quad \text{with constant difference} = 96$$
$$\{7^2, 13^2, 17^2\}, \quad \text{with constant difference} = 120$$
$$\{3^2, 15^2, 21^2\}, \quad \text{with constant difference} = 216$$
$$\{7^2, 17^2, 23^2\}, \quad \text{with constant difference} = 240$$

Fibonacci actually studied this question of squares in an arithmetic progression quite extensively, and in particular he completely characterized the common differences d that can arise from consecutive triples of perfect squares. He called these differences ***congruous numbers***, and they are described (but not defined explicitly) in the second part of Exercise 5 on page 285 of Burton. In this terminology, the objective of Exercise 5(b) is to show that every congruous number is divisible by 24. We shall give a proof of Fibonacci's result relating congruous numbers to consecutive squares in arithmetic progressions in a supplement to this unit; an interesting geometric approach to the consecutive squares question is implicit in the following article:

http://www.math.uconn.edu/~kconrad/ross2007/3squarearithprog.pdf

In either case, the consecutive squares question requires more effort than many of the number – theoretic problems discussed previously, so at this point we shall limit ourselves to proving that the common difference d is always divisible by 24; it is taken from the following online site:

http://nrich.maths.org/askedNRICH/edited/3412.html

[***Note***: This argument uses numerous concepts from courses like Mathematics 136, and it can be skipped without loss of continuity.]

Suppose that a^2, b^2, and c^2 are in an arithmetic progression whose constant difference d is not a multiple of 8. Then there is a minimal such triple such that the least number a in the progression cannot be chosen any smaller, and the middle number b is minimal for all triples a, x, y whose squares form an arithmetic progression (given a and b, the third number c is uniquely determined by the conditions of the problem).

The first step in the proof is to show that for such a minimal triple $\{a, b, c\}$ the greatest common divisors satisfy $\text{G.C.D.}(a, b) = \text{G.C.D.}(b, c) = 1$. Suppose first that there is some prime p which divides both a and b, so that $a = px$ and $b = py$ for some integers x and y. Then p^2 divides the difference $b^2 - a^2$, so that it also divides the difference $c^2 - b^2$, which in turn implies that p^2 divides c^2 and hence p divides c. If we write $c = pz$, then it follows that $\{x, y, z\}$ is a triple whose squares form an arithmetic progression with common difference $w = d/p^2$. Since d is not divisible by 8, neither is w. Thus we have constructed a new triple $\{x, y, z\}$ whose squares form an arithmetic progression such that the common difference is not divisible by 8. But we assumed that $\{a, b, c\}$ is a minimal triple of this type and $x < a$, so this is a contradiction; the source of this contradiction was our assumption that a and b had a common prime factor, and therefore this must be false. — Similar considerations imply that b and c cannot have a common prime factor.

Still assuming we have a minimal triple, we claim that a and c cannot both be even; if this were true, then $b^2 = (a^2 + c^2)/2$ would be even so a and b would share a factor of 2. This means that at most one of a, b, c is even.

Now if x is an odd integer then it is easily checked that $x^2 = 1\ (mod\ 8)$ [by one of the exercises from an earlier unit, if x is odd we know that x^2 leaves a remainder of 1 when divided by 8] and if x is even then either $x^2 = 0\ (mod\ 8)$ or $x^2 = 4\ (mod\ 8)$. [The notation means that both sides of the equation have the same remainder when divided by 8.] Therefore working mod 8 the triple $\{a^2, b^2, c^2\}$ must be one of the following:

$$(0, 1, 1)$$
$$(4, 1, 1)$$
$$(1, 0, 1)$$
$$(1, 4, 1)$$
$$(1, 1, 0)$$
$$(1, 1, 4)$$

It is clear that none of these triples are in arithmetic progression, so we have obtained a contradiction, establishing that the common difference must be a multiple of 8.

Now if the arithmetic progression has common difference $d = 1\ (mod\ 3)$ then a^2, b^2, c^2 must each have distinct [remainders or] residues ($mod\ 3$); in particular one of them must be equal to $-1\ (mod\ 3)$, which is impossible because either $x^2 = 0\ (mod\ 3)$ or $x^2 = 1\ (mod\ 3)$. Likewise, if the common difference satisfies $d = -1\ (mod\ 3)$, then the numbers a^2, b^2, c^2 must again have distinct residues ($mod\ 3$), which leads to the same contradiction. Therefore we must have $d = 0\ (mod\ 3)$, so that the common difference is a multiple of 8 and 3, which means it must be a multiple of 24.■

Further results. One can also ask whether there are even longer sequences of squares in arithmetic progressions, and a result of P. de Fermat (1601 – 1665) and L. Euler states that ***no such sequences exist.*** A proof is given in the following reference:

http://www.mathpages.com/home/kmath044/kmath044.htm

Here is a simple geometric consequence of the nonexistence result from the latter:

THEOREM. *There are no rational numbers* p, q *such that* (p^2, q^2) *is a point on the hyperbola given by*

$$(2 - x)(2 - y) = 1$$

with (p^2, q^2) *not equal to* $(1, 1)$.

Proof. Suppose we have rational numbers $p = a/b$ and $q = c/d$ (with both fractions reduced to least terms). Then if (p^2, q^2) is on the hyperbola we have

$$(2b^2 - a^2)(2d^2 - c^2) = b^2d^2$$

Since our fractions are reduced to least terms, it follows that b^2 must be relatively prime to $2b^2 - a^2$ [no common divisors except 1] and likewise c^2 must be relatively prime to $2d^2 - c^2$, so that

$$b^2 = 2d^2 - c^2 \quad \text{and} \quad d^2 = 2b^2 - a^2.$$

Rearranging terms we see that

$$b^2 - d^2 = d^2 - c^2 \quad \text{and} \quad d^2 - b^2 = b^2 - a^2.$$

Together these equations imply that a^2, b^2, d^2 and c^2 are in arithmetic progression, which we know is impossible.■

Another noteworthy achievement of Fibonacci was his solution of the cubic equation

$$x^3 + 2x^2 + 10x = 20$$

which was reportedly given to him as a challenge. His numerical approximation to the root is described on page 283 of Burton; aside from the accuracy of the result, it is worth noting how, despite his writings on the Hindu – Arabic numeration system, he (and others, including Arabic mathematicians) still wrote fractional values in the Babylonian sexagesimal notation). In his analysis of this equation he also made an important observation which foreshadowed the nineteenth century results on the impossibility of trisecting angles and duplicating cubes by means of straightedge and compass. It appears that the original version of the problem was to find a roof of the given cubic equation by means of classical Greek straightedge and compass methods. Fibonacci proved that the root could ***not*** be obtained by such methods.

Rather than attempt to give Fibonacci's proof, we shall analyze the equation from the same viewpoint we employed to study the construction problems. As in those cases, the proof that a root cannot be found using straightedge and compass depends upon showing that the polynomial $x^3 + 2x^2 + 10x - 20$ cannot be factored into a product of two rational polynomials of lower degree, or equivalently (by results of Gauss; see **http://en.wikipedia.org/wiki/Gauss%27s_lemma_(polynomial)** for details) it does not have an integral factorization of this sort. If it had such a factorization then it would have a linear factor and hence an integral root. Furthermore, if it had an integral root then this root would have to divide 20 and thus would have to be one of the following:

$$\pm 1, \quad \pm 2, \quad \pm 4, \quad \pm 5, \quad \pm 10, \quad \pm 20$$

One is then left to check that none of these twelve integers is a root of the given polynomial. Direct substitution is perhaps the most immediate way of attacking this problem, but one can also dispose of many possibilities simultaneously by noting that the polynomial under consideration is positive for all integers $x > 1$ and it is negative for all integers $x < 0$.

Fibonacci's writings on Pythagorean triples are discussed in Section 6.4 of Burton (pages 293 – 299).

Jordanus Nemorarius

One noteworthy contemporary of Fibonacci was Jordanus Nemorarius (Jordanus de Nemore, 1225 – 1260), who made contributions in several areas. He successfully analyzed mechanical problems about inclined planes that Archimedes had not been able to solve, he discovered some basic relationships between plane and spherical geometry (see addendum **(7.D)** for a description of this work), and he was the first western mathematician to use letters consistently as symbols for unknown quantities. However, aside from this innovation his mathematical terminology was rhetorical (see pages 283 – 284 of Burton).

One result of Jordanus proved a basic relationship between perfect and nonperfect numbers. Let us define a positive number to be abundant if it is less than the sum of its proper divisors and deficient if it is greater less than the sum of its proper divisors. The result of Jordanus states that a nontrivial multiple of a perfect number is abundant and a nontrivial divisor of a perfect number is deficient. In particular, this implies that every nontrivial multiple of $6, 28, \ldots$ is abundant.

Nicole Oresme

On pages 284 – 285 of Burton there is brief reference to Nicole Oresme (oh – REM, 1323 – 1382), who is widely viewed as an important figure in 14th century mathematics. Oresme made several highly original contributions to mathematics, many of which were centuries ahead of their time and some of which had a more immediate impact.

Fractional exponents. Oresme proposed a mathematically sound way of defining positive fractional powers of a number and even raised the possibility of irrational powers like **sqrt(2).**

Graphical representation of functions. The book, *Tractatus de figuratione potentiarum et mensurarum* (Latitude of Forms), written by Oresme or one of his students, popularized the idea of representing variable quantities graphically; we have already noted that the methods of Apollonius had anticipated the development of coordinate geometry much earlier, but the idea of representing variables was presented very clearly in Oresme's work, and its influence can be measured by the numerous editions of his work that were published well into the 16th century. His suggestion that certain physically measurable quantities are continuous has been implicitly assumed in many applications of mathematics to science and engineering for centuries. Oresme also speculated about graphical representations of quantities dependent on two variables by surfaces in three dimensions and possibly about analogs in even higher dimensions, but the mathematical notation at the time was inadequate.

Infinite series. During the 14th century western mathematicians began to cast aside the Greek reluctance to consider infinite processes, and in particular various infinite series were studied. It should be noted that Indian and Chinese mathematicians had studied such series much earlier, and some particularly noteworthy results of theirs from earlier times through the 15th century were rediscovered after the relevance of calculus to infinite series became apparent in the early 18th century. In particular, we have noted that the Indian mathematician Madhara (1340 – 1425) discovered the familiar infinite series for the inverse tangent function and the specialization to an infinite series for $\pi/4$, and in the next century Nilakantha Somayaji discovered a series that converges far more rapidly:

$$\frac{\pi}{4} = \frac{3}{4} + \frac{1}{3^3 - 3} - \frac{1}{5^3 - 5} + \frac{1}{7^3 - 7} - \cdots$$

Further information on this formula is given in the following paper:

> **R. Roy. *Discovery of the Series Formula for π by Leibniz, Gregory, and Nilakantha.*** Mathematics Magazine **63** (1990), 291 – 306.

Since geometric series are probably the simplest and most basic examples of infinite series, it is not surprising that medieval mathematicians were able to derive the standard formulas for such series without much trouble, and in fact they looked at numerous other problems. In Western Europe the flourishing of Scholasticism (beginning in the 12th century, and especially during the 13th and 14th centuries) was an important force behind interest in such issues, for this school of thought applied the ideas and methods of classical Greek philosophy to many new types of questions, including issues related to the notion of infinity. We shall say more about this in a subsequent unit.

Oresme used his graphical approach to provide an elegant proof for the following infinite series formula due to Robert Suiseth (or Swineshead) (*c.* 1340 – 1360), who was also known as Calculator:

$$\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots = 2.$$

A derivation of this formula by means of power series and calculus appears in the online reference **http://math.ucr.edu/~res/math9C/seriesexample.pdf**, but here we shall show how one can also prove it using basic results on rearrangements of infinite series. Specifically, consider the following tableau:

$$\begin{aligned} \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} + \cdots & \\ = \ & \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots \\ & \quad + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots \\ & \qquad + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots \\ & \qquad + \cdots \\ = \ & 1 + \frac{1}{2}(1) + \frac{1}{4}(1) + \cdots \\ = \ & 2 \end{aligned}$$

Note that in ordinary addition of finite sums, the answer does not depend upon the order or grouping of summation and that the regrouping suggested by the preceding tableau involves an infinite rearrangement and regrouping. For sums of positive quantities it is possible to justify such rearrangements, but if one is working with sums that have both positive and negative terms, then serious problems can arise. In particular, if we evaluate the infinite series for the inverse tangent of x

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

at $x = 1$ we know that the answer is $\pi/4$, but we can rearrange the terms in this series to realize an arbitrary real number as the sum; it is also possible to find rearrangements in which the new series fails to converge. A standard reference for this fact is the classic book by W. Rudin, ***Principles of Mathematical Analysis*** (3rd Ed.), pages 75 – 77.

Finally, Oresme appears to be the first person in the history of mathematics to discover that the ***harmonic series***

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \cdots + \frac{1}{n} + \cdots$$

diverges. His proof is the same one that is often seen today in textbooks: The sum of the first term by itself is $\frac{1}{2}$, the sum of the next two terms is also $\frac{1}{2}$, the sum of the next four terms is again $\frac{1}{2}$, and so on; therefore, if one adds together sufficiently many terms from this series the sum will exceed any chosen positive real number.