7.A. The closed formula for Fibonacci numbers

We shall give a derivation of the closed formula for the Fibonacci sequence F_n here. This formula is often known as ***Binet's formula*** because it was derived and published by J. Binet (1786 – 1856) in 1843. However, the same formula had been known to several prominent mathematicians — including L. Euler (1707 – 1783), D. Bernoulli (1700 – 1782) and A. De Moivre (1667 – 1754) — more than a century earlier.

We begin with general observations. Suppose we are given a sequence x_n which is defined recursively by a formula

$$x_n = bx_{n-1} + cx_{n-2}$$

where b and c are real numbers. The first point is that ***such a sequence is uniquely determined by its initial values*** x_0 and x_1.

<u>Proof.</u> Suppose that x_n and y_n are two such sequences with values of P and Q when $n = 0$ or 1. Then $x_n = y_n$ when $n = 0$ or 1; assume that the values of the sequence are equal for all $n < k$, where $k > 1$. Then we have

$$x_k = bx_{k-1} + cx_{k-2} = by_{k-1} + cy_{k-2} = y_k$$

and therefore the two sequences are equal by mathematical induction.

In favorable cases one can write down the sequence x_n in a simple and explicit form. Here is the key step which also applies to a wide range of similar problems.

<u>PROPOSITION.</u> *Suppose that r and s are distinct roots of the **<u>auxiliary polynomial</u>** $g(t) = t^2 - bt - c$. Then for every pair of constants u, v the sequence $ur^n + vs^n$ solves the **<u>finite linear difference equation</u>** $x_n = bx_{n-1} + cx_{n-2}$.*

<u>Derivation.</u> Let $y_n = ur^n + vs^n$; we need to show that

$$y_n - by_{n-1} - cy_{n-2} = 0$$

for all $n > 1$. If we expand the left hand side and we obtain the following equations.

$$(ur^n + vs^n) - b(ur^{n-1} + vs^{n-1}) - c(ur^{n-2} + vs^{n-2}) =$$

$$u(r^n - br^{n-1} - cr^{n-2}) + v(s^n - bs^{n-1} - cs^{n-2}) =$$

$$ur^{n-2}g(r) + vs^{n-2}g(s) = ur^{n-2}\cdot 0 + vs^{n-2}\cdot 0 = 0.$$

Therefore $x_n = ur^n + vs^n$ solves the original equation.

One can take this further to find the unique solutions satisfying $x_0 = P$ and $x_1 = Q$ by solving the equations $u + v = P$ and $ur + vs = Q$ for u and v. It is always possible to find a unique solution to this system of two linear equations because r and s are distinct (the general formula for u and v in terms of P, Q, r, and s can be derived fairly easily).■

We shall now apply all this to the Fibonacci equation

$$F_n \quad = \quad F_{n-1} \ + \ F_{n--2}$$

whose auxiliary polynomial is equal to

$$x^2 = x + 1.$$

Equivalently, one can write this polynomial in the form

$$x^2 - x - 1 = 0$$

and since the roots of this equation are

$$\phi = \frac{1+\sqrt{5}}{2}, \qquad \psi = \frac{1-\sqrt{5}}{2}$$

it follow that the closed formula for the Fibonacci sequence must be of the form

$$f_n = u\phi^n + v\psi^n$$

for some constants u and v. If we now use the conditions $F_0 = 0$ and $F_1 = 1$, we see that

$$0 = u\phi^0 + v\psi^0, \qquad 1 = u\phi^1 + v\psi^1$$

where the first equation simplifies to $u = -v$; substituting this into the second one yields

$$1 = u\left(\frac{1+\sqrt{5}}{2}\right) - u\left(\frac{1-\sqrt{5}}{2}\right) = u\left(\frac{2\sqrt{5}}{2}\right) = u\sqrt{5}.$$

Therefore

$$u = \frac{1}{\sqrt{5}}, \qquad v = \frac{-1}{\sqrt{5}}$$

and accordingly we have

$$f_n = \frac{\phi^n}{\sqrt{5}} - \frac{\psi^n}{\sqrt{5}} = \frac{\phi^n - \psi^n}{\sqrt{5}}. \blacksquare$$

<u>Comments on difference equations.</u> Finite difference equations are analogous to ordinary differential equations in several respects; for example, they can be used to analyze processes involving discrete changes in much the same way that ordinary differential equations are used to analyze processes involving continuous changes. One everyday example of a discrete process is an ***<u>amortized loan</u>***, in which a borrower agrees to pay back a loan of L dollars, at an annual simple interest rate of R percent, in N monthly payments of P dollars each; the basic condition of such a loan is that, in each payment, ***interest is only paid on the remaining balance.*** It turns out that computing P in terms of the other variables reduces to the solution of a linear first order difference equation. A derivation of the formula (using linear algebra) is given on pages 13 – 14 of the file **http://math.ucr.edu/~res/math132/linalgnotes.pdf**.