14. Calculus after Newton and Leibniz

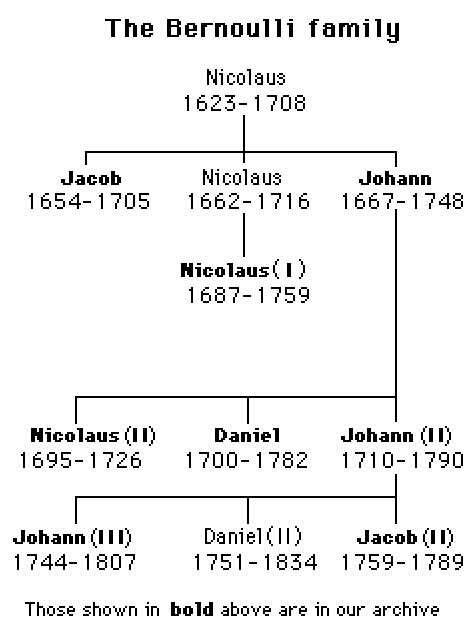
(Burton, **9.3, 10.2, 11.3**)

> The calculus grew organically, sending forth branches while simultaneously putting down firm roots. The roots were the subject of philosophical speculation that eventually led to new mathematics as well, but the branches were natural outgrowths of pure mathematics that appeared very early in the history of the subject.
>
> R. Cooke, ***The History of Mathematics***, page 475

We shall concentrate on points related to the material covered in first year calculus courses. Mathematics has continued to grow rapidly during the 300 + years following the invention of differential and integral calculus by Newton and Leibniz, but most of this history is well beyond the scope of the present course.

The Bernoulli family

Since several members of this family made important contributions to the mathematical and physical sciences beginning in the late 17th century and continuing throughout the 18th century, we shall insert a family tree which includes those family members whose mathematical achievements were highly notable; certain other descendants are also well – known, and one of them is the 1946 Nobel Prize winning German – Swiss novelist H. Hesse (1877–1962), who is known for books such as ***Steppenwolf*** and ***Siddhartha***.



http://www-groups.dcs.st-and.ac.uk/~history/Diagrams/Bernoulli_family.gif

The first Bernoulli brothers — James or Jacob or Jakob or Jacques (1655 – 1705) and John or Johann or Jean (1667 – 1748) — made numerous important contributions to calculus and its applications soon after calculus was developed and became known to other scientists and mathematicians. One small item worth noting is that the result known as ***L'Hospital's Rule*** was originally due to John Bernoulli but was sold to G. de L'Hospital (1661 – 1704) for an influential textbook the latter published in 1696. The Bernoullis were particularly effective at applying the methods of differential and integral calculus to analyze new types of mathematical questions that had previously been out of reach. These include the properties of special algebraic and nonalgebraic curves, infinite series, optimization problems, and the solution of ***separable differential equations*** that can be expressed in the form $y' = P(x)\,Q(y)$ for suitable functions $P(x)$ and $Q(y)$. Two specific examples of problems they promoted were the derivation of the equation for the ***catenary*** or "hanging chain curve" (given by the graph of $f(x) = a\cosh bx$ for suitable positive constants a and b) and the ***brachistochrone problem***, which asks for the curve of quickest descent connecting two given points in a vertical plane; it turns out that a portion of the cycloid curve is a solution to this question. Several members of the Bernoulli family made several other early contributions to the study of differential equations and mathematical probability theory, and they also discovered several other extremely important applications of calculus and differential equations to physics. The areas of physics they studied include optics, astronomy, fluid mechanics, wave motion, heat conduction and elasticity.

Some of the mathematical disputes involving members of the family are mentioned on pages 474 – 475 of Burton. There was also a later dispute about work between John and Daniel about separate publications on hydrodynamics in 1738; subsequent analysis of historical records does not support the accusations raised by John Bernoulli in this case.

Solid analytic geometry

When plane analytic geometry was developed during the 17th century, several researchers including like Fermat, Schooten and P. de la Hire (1640 – 1718) were convinced that one could handle questions in $3-$ dimensional geometry similarly by adding one more coordinate and making suitable adjustments to various formulas, but the details were not fully worked out until the 18th century. Names associated with this work include J. Hermann (1678 – 1733), A. – C. Clairaut (1713 – 1765) and L. Euler.

Coordinate geometry also allows one to study geometrical questions in dimensions greater than 3; for example, to study $4-$ dimensional geometry we view points as given by ordered quadruples of real numbers. In the 18th century J. le Rond d'Alembert (1717 –1783) suggested that time could be considered as a fourth dimension, and in the 19th century mathematicians began to study higher – dimensional geometry both for its own sake and for its physical applications (for example, there are numerous places in classical mechanics where it is useful to use a model corresponding to a space of 4 or even more dimensions). There is a discussion of this topic at an advanced undergraduate level in Chapter 16 (pages 216 – 230) of the following book by a well – known author of books on science for a popular audience:

> **I. N. Stewart. *Taming the infinite: The Story of Mathematics from Babylonian Numerals to Chaos Theory.*** Quercus Publishing, London, 2008.

Infinite series

We note first that the standard infinite power series for functions are named after B. Taylor (1685 – 1731) and C. Maclaurin (1698 – 1746); the naming of the power series

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots.$$

for Maclaurin is an accident of history; series of this type were recognized well before Maclaurin discussed them in one of his books, and he never claimed credit for discovering them.

First year calculus books almost always mention that the infinite series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

converges, but often the value of the sum is not mentioned. In fact, Euler proved the unexpected relationship

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$$

using elaborate manipulations of infinite series. This was just one in a sequence of increasingly bold and dramatic summation formulas that Euler derived.

Such conclusions ultimately led Euler to carry out many speculative operations on infinite series that do **not** converge. In particular, he suggested that ½ is a reasonable value to take as the sum of the following divergent series:

$$1 + (-1) + 1 + (-1) + 1 + (-1) + 1 + (-1) + \ldots$$

This may seem absurd, but it turns out that one can find some good mathematical justifications for attaching this value to the divergent series. Examples of this sort are somewhat artificial, but one can also construct important classes of physical problems involving infinite series with closely related convergence difficulties. Predictably, one eventually encounters supposed formulas that lead to bizarre and sometimes contradictory answers. One particularly striking example was Euler's "conclusion"

$$-1 = 1 + 2 + 3 + 4 + \ldots$$

which is discussed on page 534 of Burton, and a similar "identity" is discussed in **(14.A).**

However, in many cases it turns out that there are fairly large classes of infinite series which are divergent in the usual sense but can be viewed as "weakly convergent" in some precisely generalized sense. One particularly important example is summability in the sense of E. Cesàro (1859 – 1906), and in fact Cesàro's summation method yields Euler's "reasonable value" of ½ for the sum displayed above. Euler has been criticized for mindless formal manipulation in connection with his attempts to sum divergent series, and his view that every divergent series could be somehow evaluated was at best far too optimistic, but it is important to remember that he explicitly recognized the highly speculative nature of his thoughts on this issue, and one of his objectives was to see how far mathematicians could push the methods that proved to be so useful and reliable in other contexts (in other words, ***he was testing the robustness of the methods).*** These issues are discussed further on pages 526 – 528 of the following article:

V. S. Varadarajan. ***Euler and his work on infinite series.*** Bulletin of the American Mathematical Society (2) **44** (2007), 515 – 539.

Euler's contributions to mathematics

Euler was the most prominent mathematician of the 18th century, and one of the most productive and influential mathematicians of all time; in particular, new publications of his continued to appear until 1831 (48 years after his death). For many topics, his approaches are still the definitive treatments. There is a discussion of Euler on pages 527 – 537 of Burton which mentions some aspects of his life and work; we shall include some additional information and comments.

In addition to his introduction of i to denote the square root of -1 and his popularization of using π to denote the ratio of the circumference of a circle to its diameter, Euler is also responsible for numerous other common notational conventions. These include the familiar symbolism $f(x)$ to denote the value of the function f for the variable x, the modern notation for the trigonometric functions, the letter e for the base of the natural logarithms, and the large Greek letter Σ to denote "the summation of all terms with the specified form." The basic formula

$$e^{i\theta} \;=\; \cos\theta + i\cdot\sin\theta$$

is part of his definition of exponential and logarithmic functions for arbitrary complex numbers. We have already mentioned some of his contributions to number theory. In his work on this subject, Euler was one of the first to employ methods from calculus, and as such his work anticipated the discovery of deep relations between calculus and number theory beginning in the 19th century (see supplement **(14.E)** for more on this topic; the proof of Fermat's Last Theorem relies strongly on such relationships). Euler also did extensive and extremely substantial work on developing and applying the methods of calculus to physical problems and to discrete mathematics (in particular, graph theory), but these contributions are beyond the scope of this course. The so – called ***Euler Formula*** for polyhedra (which was also known to Descartes) has already been mentioned in **http://math.ucr.edu/~res/math153/history03d.pdf**.

Page 529 of Burton retells a frequently repeated story about a supposed encounter between Euler (who held strong and fairly traditional religious beliefs) and the prominent French encyclopedist D. Diderot (1713 – 1784), who was harshly critical of religious orthodoxy. Burton gives the standard reference for the story, asserting it "has circulated for so many years that historians tend to give it some credence," but he also cast doubts on the details of the story. There is an even more critical view of the story in a footnote on page 128 of Struik's ***Concise History of Mathematics***:

> This is a good example of a bad historical anecdote, since the value of an anecdote about an historical person lies in its faculty to illustrate certain aspects of his character; this particular anecdote serves to obscure the character both of Diderot and of Euler. Diderot knew his mathematics and had written on involutes [*a class of plane curves derived from other examples*] and probability, and no reason exists to think that the thoughtful Euler would have behaved in the asinine way indicated.

In connection with the Diderot – Euler anecdote, it is also worthwhile to note that Catherine the Great (1729 – 96, Empress of Russia 1762 – 96) continued to support Diderot financially following his return to France from St. Petersburg.

Trigonometric series

The idea that sounds are caused by vibrating objects or media dates back to the Pythagoreans, if not earlier, and Leonardo Da Vinci's experiments around the year 1500 confirmed that sound is propagated by vibrating waves. One of Galileo's many experimental achievements was to demonstrate that the basic tone or pitch of a sound depended upon the fundamental frequency of a periodic vibration, and in the 18^{th} century D. Bernoulli observed that the shading differences between two sounds with a given fundamental frequency correspond to different combinations of ***harmonic overtones***; in other words, there is vibration at a basic frequency which is modified by smaller vibrations at some (integral) multiple of the basic frequency. As noted in supplement **(14.F),** it follows that the waveform of a sound is given by a trigonometric series given by linear combinations of the functions $\sin(2\pi nx/P)$ and $\cos(2\pi nx/P)$, where n is a nonnegative integer. Several examples are also discussed in **(14.F).**

The existence and usefulness of such trigonometric expansions require some understanding of the following question:

If we are given a function $f(x)$ which is periodic with some positive period P — algebraically, $f(x+P) = f(x)$ for all values of x — to what extent is it possible to express f as a series whose terms are "pure vibrations" which are either constant or given by one of the functions

$$b_n \sin\left(\frac{2\pi nx}{P}\right) \quad , \quad a_n \cos\left(\frac{2\pi nx}{P}\right) \qquad \textit{where } n \textit{ is some positive integer}\textbf{?}$$

This question is covered in some but not all introductory calculus textbooks or supplementary online material for such books. The following is a typical example:

http://www.stewartcalculus.com/data/CALCULUS%20Early%20Transcendentals/upfiles/FourierSeries5ET.pdf

To simplify the discussion, we shall restrict attention to the case where the period is 2π; the general case can be retrieved by a simple change of variables.

If the function f is a ***finite sum*** of the types of functions described above, then some standard integral calculations, which appear in nearly every textbook covering real variables or boundary value problems in partial differential equations, show that the coefficients in the expansion

$$\frac{a_0}{2} + \sum_{n=1}^{N} [a_n \cos(nx) + b_n \sin(nx)]$$

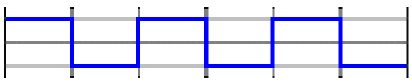
are given as follows:

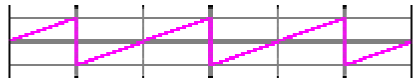
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

Note that by periodicity we could equally well integrate between 0 and 2π. In his boldly original work on heat conduction (*Théorie analytique de la chaleur*), J. – B. J. Fourier (1768 – 1830) raised the issue of expanding fairly arbitrary periodic functions using infinite ***Fourier series*** of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

where the coefficients a_n and b_n are defined by integral formulas as above. One extremely provocative feature of his theory was to include the possibility of representing discontinuous functions by such series. There is good physical motivation for this because the ***square wave*** and ***sawtooth wave*** functions with graphs





(Source: http://en.wikipedia.org/wiki/Square_wave)

model important types of physical vibrations (for example, square wave vibrations arise when one turns up an amplifier so high that the tone gets seriously distorted). Immediate questions about the convergence of such series were soon justified as mathematicians discovered examples where these trigonometric series do not behave as nicely as the power series that had been used systematically and reliably for some time, even in cases where one starts with a continuous function. However, it also became clear that these series had some theoretical and experimental legitimacy even for discontinuous examples like those described above, and one important problem in the 19^{th} and early 20^{th} centuries was to understand the convergence properties of such series in greater depth. A detailed discussion of results on this question is beyond the scope of this course, but there is a summary (written at the beginning graduate level) in the online reference

http://en.wikipedia.org/wiki/Convergence_of_Fourier_series

and for our purposes the main point is that trigonometric series were an important reason for mathematicians to take a much closer look at the logical soundness of infinite series expressions.

There is a more detailed account of Fourier's life and work on pages 610 – 614 of Burton.

Finally, here are some standard additional online references for Fourier series:

http://en.wikipedia.org/wiki/Fourier_series

http://mathworld.wolfram.com/FourierSeries.html

http://www.physics.miami.edu/~nearing/mathmethods/fourier_series.pdf

http://www.uwec.edu/walkerjs/media/fseries.pdf

http://www.intmath.com/Fourier-series/Fourier-intro.php

The following extraordinary and extremely accessible book on the subject is also very highly recommended:

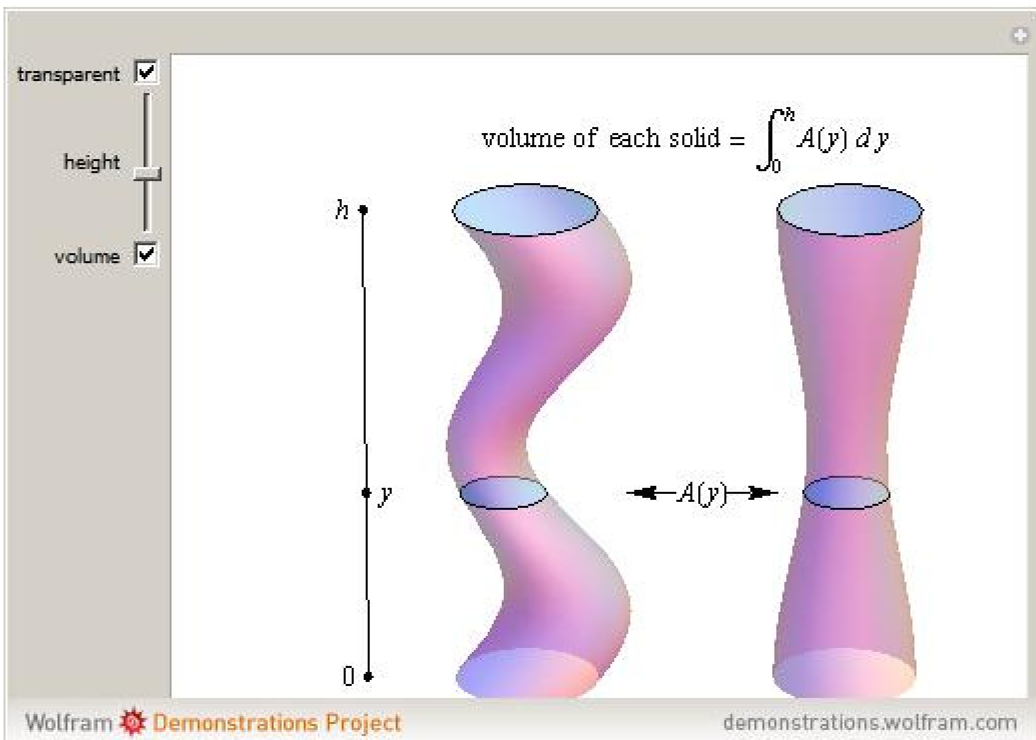
Transnational College of LEX. *Who Is Fourier?: A Mathematical Adventure*. Language Research Foundation, Tokyo, Japan, 1995.

The preceding discussions lead naturally to the final topic:

The logical soundness of calculus

Aside from the questions on infinite series that we have just raised, there are even more important issues regarding calculus that were problematic during the 17th and 18th century. The most important of these was the use of ***infinitesimals***. As indicated in Unit 12, the idea is well illustrated in the method employed by Cavalieri to study the volume of a solid **A** that is contained between two parallel planes. If we assume that the planes are defined by the equations $z = 0$ and $z = 1$, then for each t between 0 and 1 there is the cross section $\mathbf{A}_t$ formed by Intersecting **A** with the parallel plane defined by $z = t$. Cavalieri's idea was to view **A** as composed of an infinite collection of cylindrical solids whose bases are the cross sections $\mathbf{A}_t$ and whose heights are some very small, in fact ***infinitesimally small***, value that we shall call dt. From this viewpoint, the total volume is obtained by adding the volumes of these infinitesimally short cylindrical solids; in modern terminology, one adds or integrates these infinitesimals by taking the definite integral of the area function with respect to t from 0 to 1. Of course, the point of this discussion is to suggest that the volume of **A** is given by the following standard integral formula in which $a(t)$ denotes the area of the planar section $\mathbf{A}_t$:

$$V = \int_0^1 a(t)\,dt$$



(***Source:*** http://demonstrations.wolfram.com/CavalierisPrinciple/HTMLImages/index.en/popup_3.jpg)

Similar considerations apply to differentiation. To approximate the slope of the tangent line to a curve $y = f(x)$, one looks at the slope of the secant line joining (x, y) to $(x + \Delta x, y + \Delta y)$, where $\Delta y = f(x + \Delta x) - f(x)$. This slope is equal to the quotient $\Delta y/\Delta x$, and the idea is that the slope of the tangent line is the corresponding quotient when Δx is the infinitesimally small quantity dx, in which case Δy becomes the associated infinitesimally small quantity dy.

It was clear to 17th and 18th century scientists and philosophers that such infinitesimals were supposed to be smaller than any finite quantity but were still supposed to be positive. Not surprisingly, there were many questions about the logical consistency of using objects that were smaller than any finite positive quantity but still positive. If one is careless with such a notion it is easy to contradict the principle that between any two real numbers there is a rational number; a crucial question is *whether it is ever possible to be careful enough to avoid these or other logical difficulties*. Some of these issues lead directly to the sorts of paradoxes that Zeno had formulated more than 2000 years earlier. Although proponents of calculus made vigorous and repeated efforts to explain infinitesimals and the computational methods of Newton and Leibniz were yielding highly reliable answers, these explanations did not really clarify the situation to some mathematicians or others of that era.

Probably the most famous critique of infinitesimals was ***The Analyst***, by (Irish – Anglican) Bishop G. Berkeley (BARK – lee, 1685 – 1753); mathematicians and others realized the validity of his logical objections (as usual, it is beyond the scope of this course to assess his philosophical conclusions). Progress in mathematics continued at a rapid pace, but Berkeley's criticisms reinforced earlier views of many that calculus needed a more secure logical foundation. With the development of calculus, mathematics had moved into new territory, not just abstracting familiar ideas but also contributing new concepts of its own. It was also rapidly accepting an ever expanding collection of ideas and methods that were increasingly removed from ordinary experience. *In order to handle such new concepts it is necessary to maintain very strict and abstract logical standards which compensate for the increased remoteness from sensory experience.*

Towards the end of the 18th century Lagrange expressed strong dissatisfaction with the logical justification for calculus, and he proposed that the subject be formulated using infinite series. However, this did not suffice, for as noted above *there were also serious questions about the elaborate manipulations with infinite series that mathematicians had been performing.* The crucial step towards resolving these difficulties was the reformulation of calculus using the concept of ***limit*** (a concept which had not appeared explicitly in the work of Newton or Leibniz). The potential usefulness of such a notion had already been tentatively anticipated by Wallis and Gregory; in fact, d'Alembert had already proposed a definition of limits, but the wording needed to be made more precise. There are somewhat different versions of this concept for continuous functions and sequences, and of course infinite series can be discussed using the second version because the simplest way to define them it to take the limit of the finite partial sums. Clear and usable concepts of such limits were (independently) due to J. A. Da Cunha (1744 – 1787), B. Bolzano (1781 – 1848), and A. – L. Cauchy (1789 – 1857). The difference can be described as follows: Instead of working with a single infinitesimally small quantity, one thinks instead of an *extremely small variable quantity which is rapidly shrinking and is eventually less than any fixed finite quantity.*

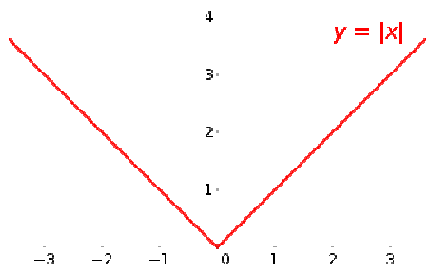
Bolzano's work is particularly noteworthy for anticipating many important features in the ultimate development of the foundations for calculus. In particular, he understood the importance of the ***completeness property*** for the real number system (an infinite series of nonnegative terms has a finite sum if and only if there is a uniform upper bound for its partial sums), he recognized the need to prove the ***Intermediate Value Property*** for continuous functions and proved it within his framework, and he foresaw some basic results in set theory (for example, the set of real numbers has a larger **"order of infinity"** than the infinite set of all nonnegative integers). For a variety of reasons, Cauchy's work had the greatest impact. In particular, his textbook of 1821 describes the concept of limit in a form very close to the one in use today, and his definition of derivative is precisely the one used today. Cauchy also stressed that the definite integral should be defined as the limit certain algebraic sums and is independent of the definition of the derivative. It is from Cauchy's view of the integral that broad modern generalizations of this concept have developed. Cauchy's work also contains many of the straightforward convergence tests for infinite series which have been in calculus textbooks ever since his time, but as noted below his results left some convergence questions unresolved, particularly when one also takes into account the work of Fourier. Here are some of the main questions that still required answers:

1. How does one formulate the intuitively clear ideas in, say, Cauchy's formulations with more precise and quantitative language? The first full paragraph on page 614 of Burton (beginning on line 8) contains some comments about this issue.
2. What can be said about the convergence of trigonometric and other familiar series?
3. In connection with both of the preceding questions, what is a suitably precise and useful way of defining the concept of a function of a real variable?
4. In connection with the preceding question, what is a suitably precise and useful way of describing the real number system?

Several mathematicians found answers to these questions during the 19th century. In particular, the modern quantitative definition of limit in terms of two positive numbers δ (*delta*) and ε (*epsilon*) is due to K. T. W. von Weierstrass (1815 – 1897), the modern definition of continuity combines this with Cauchy's formulation, and the approach to defining integrals in most modern calculus textbooks is due to G. F. B. Riemann (1826 – 1866). For a very wide range of present day mathematical purposes, the definitive formulation of integral is a modification of Riemann's which is due to H. Lebesgue (pronounced le – BAYG, 1875 – 1941). There are more detailed discussions of both Weierstrass and Riemann on pages 614 – 619 and 723 – 724 of Burton, and there is a more detailed treament of the Riemann integral on pages 726 – 727 of ***History of Mathematics – An Introduction*** (2nd Edition), by Katz. These improved formulations of limits and continuity also led to a logically rigorous proof of another key result about continuous functions; namely, if a continuous function is defined on a **closed** interval, then it takes maximum and minimum values on that interval (the result clearly fails for intervals which do not contain both endpoints or are infinite in at least one direction). The earliest published proof of this result was due to E. Heine (1821 – 1881; not to be confused with the well – known German language poet H. Heine, 1797 – 1856).

The writings of N. H. Abel (1802 –1831) played an important part in the critical analysis of convergence questions for infinite series, which had not received much attention (except for the geometric series) until early in the 19th century. One of the first serious steps in this direction was a proof by Gauss that a specific new example (his ***hypergeometric*** function) converges; this function is defined in **http://planetmath.org/encyclopedia/HypergeometricFunction.html** with extensive further information at **http://mathworld.wolfram.com/HypergeometricFunction.html**. As noted on page 610 of Burton, Abel noted that a convergence assertion of Cauchy's was false for some trigonometric series, and clearly this was tied to the more general convergence questions arising from Fourier's work. This increased attention to convergence caused many mathematicians including P. – S. de Laplace (1749 – 1827) to re – examine their use of infinite series and confirm that the formal manipulations in their papers were justified. In 1829 P. G. Lejeune Dirichlet (1805 – 1859) made a very important advance, proving a fairly weak sufficient condition for the convergence of Fourier series which applied to a fairly wide class of periodic functions including (i) all functions which could be written as the difference of two monotonically nondecreasing functions over the periodic interval except at the interval's endpoints, (ii) all functions which have continuous derivatives. This was reassuring, but in the course of studying this problem, mathematicians also discovered that both continuous and discontinuous functions could behave in bizarre manners that they had not previously imagined, and for several reasons it was absolutely necessary to take such examples into account. Some particularly significant examples were due to Bolzano, Dirichlet and Weierstrass. On the other hand, results of Cauchy, Weierstrass and S. Kovalevskaya (1850 – 1891) showed that, for the most part, the infinite series expansions for functions solving basic questions in physics did converge in a very good manner. The previously cited Wikipedia link on the convergence of Fourier series contains additional information on the convergence or non – convergence of these series.

One question arising immediately from Fourier's work was to find a sufficiently broad, precise and usable definition for a function. Several mathematicians formulated tentative definitions during the 18th century, but in some cases they were imprecise and in others their insistence on nice, explicit analytic formulas excluded fundamental examples like the absolute value function (as usual, $|x|$ equals x if x is nonnegative and $-x$ if x is nonpositive).



(***Source:*** http://upload.wikimedia.org/wikipedia/commons/thumb/6/6b/Absolute_value.svg/360px-Absolute_value.svg.png)

The decisive step towards defining a mathematical function was due to Dirichlet, who defined a function as an arbitrary rule which assigns to each value of some ***independent variable*** an associated value of the ***dependent variable***. This definition is discussed and stated completely on pages 613 – 614 of Burton. The modern mathematical definition of function in is basically just a direct translation of Dirichlet's definition into set – theoretic language.

This was not quite the end of the story; functions were defined in terms of variables, but *it was also necessary to be more precise about exactly how these variables — real numbers — could be described in a precise mathematical sense.* Ultimately mathematicians realized that a secure logical foundation for calculus required a logically rigorous description of the real number system and a firm understanding of the real numbers themselves; these in turn required a ***theory of infinite sets***. Both developments were completed during the second half of the 19th century. In order to describe the breakthrough for describing real numbers, it is useful to go back to the Greek discovery that some numbers were irrational. After reaching this conclusion, Greek mathematicians turned to geometry as a foundation for mathematics precisely because their understanding of irrational numbers was incomplete. However, the work of Eudoxus of Cnidus (*c.* 408 – 355 B. C. E.) yielded one very important property of real numbers; namely, *between any two real numbers there is a rational number.* By the end of the 16th century our usual understanding of real numbers in terms of infinite decimals was a well established principle in European mathematics, science and engineering. While this was sufficient for all computational purposes, it did not shed much conceptual light on questions about continuity, the existence of limits, and the overall logical soundness of the concept of a real number. The final insights in the process were mainly and independently due to R. Dedekind (1831 – 1916) and

G. Cantor (1845 – 1918), and it was essentially a converse to the principle implicitly due to Eudoxus: Specifically, the real numbers are in some sense the *absolutely largest possible number system* in which everything can be approximated by rational number to any desired degree of accuracy. *Justifying this viewpoint in a logically rigorous manner requires the methods and results of set theory* in the form developed by Cantor and subsequent researchers.

Additional information on the historical development of set theory is presented on pages 4 – 14 of the following online reference:

http://math.ucr.edu/~res/math144/setsnotes1.pdf

Still more information (but at a more advanced level) appears in Sections **III.4** and **VII.5** of the following related references:

http://math.ucr.edu/~res/math144/setsnotes3.pdf

http://math.ucr.edu/~res/math144/setsnotes7.pdf

A logically rigorous approach to infinitesimals. Despite the doubtful logical status of infinitesimals and the development of a rigorous foundation for calculus built upon set theory and the concept of limit, many subsequent users of mathematics continued to work with infinitesimals, even well after mathematicians had discarded the latter (for foundational purposes) in the 19th century. Probable motivations include their relative simplicity, the fact that they were giving reliable answers, and an expectation that mathematicians would ultimately find a logically rigorous justification for whatever was being attempted. Similarly, even after the creation of a sound logical foundation for calculus the use of infinitesimals in calculus and other textbooks continued well into the 20th century for several reasons, including their continued and frequent use in science and engineering. The following textbook, which was widely used and went through many editions over more than half a century, is a typical example:

> **W. A. Granville, P. F. Smith and W. R. Longley, *Elements of Differential and Integral Calculus*** (Various editions from 1904 to 1962). Wiley, New York, 1962. **ISBN:** 0–471–00206–2.

Eventually mathematicians did succeed in giving a logically rigorous justification for the concept of infinitesimals; in particular, during the 1960s A. Robinson (1918 – 1974) used extensive machinery from abstract mathematical logic to show that one can in fact construct a number system with infinitesimals that satisfy the usual rules of arithmetic. However, the crucial advantage of Robinson's modern concept of infinitesimal — its logical soundness — is seriously offset by the fact that, unlike 17th century infinitesimals, it is neither simple nor intuitively easy to understand. The associated theory of ***Nonstandard Analysis*** has been studied to a considerable extent mathematically, but it is not widely used in the traditional applications of the subject to the sciences and engineering, and within mathematics it has not replaced the framework for calculus which was developed during the 19th century; on the other hand, some recent work in mathematical economics has been formulated within the context of nonstandard analysis. The following online references provide further information on this subject:

http://members.tripod.com/PhilipApps/nonstandard.html

http://www.haverford.edu/math/wdavidon/NonStd.html

http://mathforum.org/dr.math/faq/analysis_hyperreals.html

http://en.wikipedia.org/wiki/Nonstandard_analysis

http://www.math.uiuc.edu/~henson/papers/basics.pdf

http://en.wikipedia.org/wiki/Criticism_of_non-standard_analysis

Here are a few textbook references for nonstandard analysis**:**

J. M. Henle and E. M. Kleinberg, ***Infinitesimal Calculus.*** Dover Publications**,** New York**,** 2003**.**

J. L. Bell, ***A Primer of Infinitesimal Analysis*.** Cambridge University Press**,** New York**,** 1998.

A. E. Hurd and P. A. Loeb, ***An Introduction to Nonstandard Real Analysis*** (Pure and Applied Mathematics**,** Vol. 118)***.*** Academic Press, Orlando**,** FL**,** 1965**.**

There is also a complete elementary calculus textbook online which uses the methods of nonstandard analysis**:** **http://www.math.wisc.edu/~keisler/calc.html**