

## 14.A. An infinite series fallacy

Here is a particularly striking example of manipulations with infinite series that yield absurd conclusions. This example is due to Euler. Bibliographic information and further examples are discussed on pages 444–449 of Kline, *mathematical Thought from Ancient to Modern Times*.

We start with the geometric series:

$$\frac{x}{1-x} = x + x^2 + x^3 + \dots$$

On the other hand, consider also the following series expansion:

$$\frac{x}{x-1} = \frac{1}{1-\frac{1}{x}} = 1 + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \dots$$

If we add these two equations we obtain the purported equation

$$0 = \frac{x}{1-x} + \frac{x}{x-1} = \dots + \frac{1}{x^3} + \frac{1}{x^2} + \frac{1}{x} + 1 + x + x^2 + x^3 + \dots$$

which should look suspicious for several reasons. For example, if  $x = 2$  or  $x = \frac{1}{2}$  then we obtain the purported identity

$$0 = \dots + \frac{1}{2^3} + \frac{1}{2^2} + \frac{1}{2} + 1 + 2 + 2^2 + 2^3 + \dots$$

which seems to suggest that an infinite sum of certain positive real numbers is zero!

What is wrong here? If we substitute  $x = \frac{1}{2}$  into the first geometric series then we certainly get a valid result. However, the second expansion is definitely not valid if  $x = \frac{1}{2}$ . The reason for this failure is simple: The terms of the infinite series expansion do not go to zero as  $n \rightarrow \infty$ . ■