14.C. Nonelementary indefinite integrals

In both the theory and applications of calculus, one central feature is that many of the functions which are described using the basic constructions — arithmetic operations, taking n^{th} roots, taking exponentials or logarithms, taking trigonometric functions and their inverses — have indefinite integrals (or antiderivatives) of the same type. Calculus textbooks give an assortment of standard techniques for integrating such functions, including change of variables, integration by parts, partial fraction expansions or trigonometric substitutions.

However, there are also many examples of such functions for which these techniques break down and the indefinitie integral cannot be described using the basic constructions mentioned above. One particularly noteworthy example is the function

$$f(x) \quad = \quad \exp(ax^2)$$

(where a is an arbitrary nonzero number), and the purpose of this document is to provide some further information and references concerning this fact.

The meaning of mathematical impossibility

Everyone knows of examples where things previously thought to be impossible were eventually done, and it is natural to challenge such an impossibility assertion; even though no one has found a way to write the indefinite integral of $\exp(ax^2)$ in terms of the usual elementary functions in calculus, it may seem plausible that someone will eventually be able to do so. However, the question here is much narrower than asking whether someone can come up with a method for evaluating an integral whose integrand is equal to such a function; we want to do so using a function of a specific type. To illustrate this in a simpler context, consider the following statement:

It is mathematically impossible to express the indefinite integral of the cosine function as a polynomial function.

In the displayed statement, the problem is not that no one has been clever enough to find the right polynomial, but rather that ***no such polynomial can exist.*** For suppose that one could be found. Then its derivative would be the cosine function, but it would also be a polynomial. We know that the 4^{th}, 8^{th}, 12^{th}, *etc.* derivatives of the cosine function are equal to the cosine function itself, and thus infinitely many higher order derivatives are nonzero. In contrast, only finitely many higher derivatives of a polynomial are zero. Thus when we say that something of this sort is mathematically impossible, it does not reflect a sense of defeatism, but rather a convincing demonstration that any potential solution would lead to conclusions that are logically inconsistent.

The normal distribution properties of $\exp(-ax^2)$

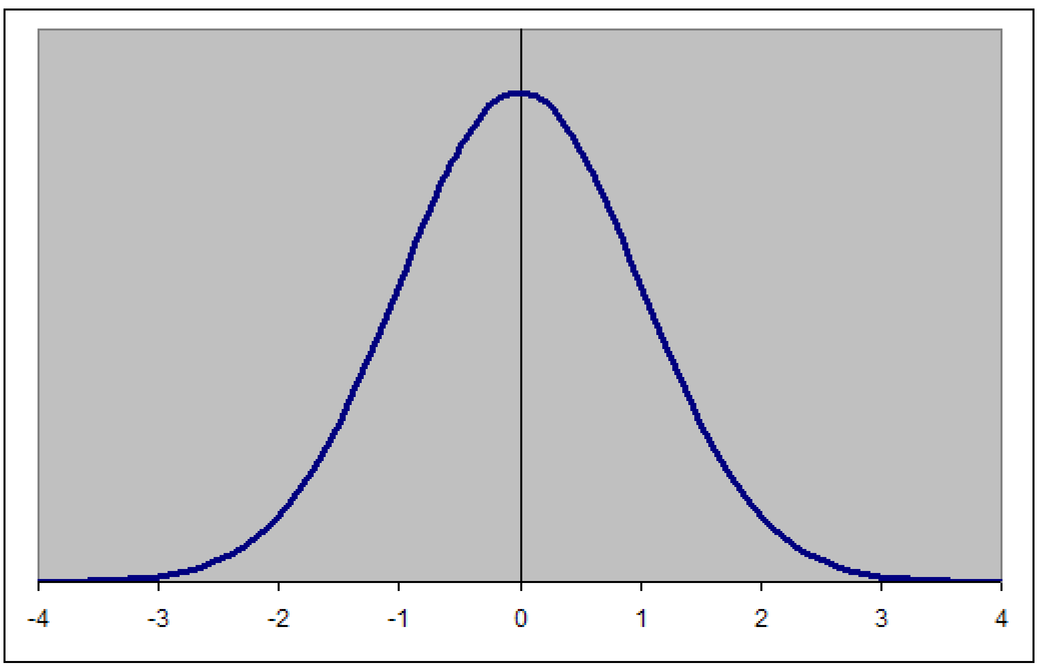
One reason for interest in the function described above is its relevance to probability theory and statistics. The function $\exp(-ax^2)$ is positive valued for all real numbers x, and if a is positive it is fairly easy to prove that the improper integral of this function from 0 to ∞ converges by a comparison with $\exp(-ax)$, whose improper integral over the same values is easily checked to converge; more precisely, we have the inequality $\exp(-ax^2) < \exp(-ax)$ if $x > 1$, and the integral of the latter from 0 to ∞ equals $1/a$. In fact, one can evaluate the improper integral of $\exp(-ax^2)$ explicitly using methods from multivariable calculus:

$$\int_0^\infty e^{-ax^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

The appearance of π in this formula suggests an unexpected significance of π for probability theory that we shall not try to explain here. It follows immediately that if we modify the original function to obtain

$$f(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

then the integral of the latter from $-\infty$ to $+\infty$ is equal to 1. If we graph this function we obtain the following bell shaped curve which is fundamentally important in probability and statistics.



(***source:*** http://www.tushar-mehta.com/excel/charts/normal_distribution/)

The proof of mathematical impossibility

The first rigorous mathematical proof that the integral of $\exp(-ax^2)$ is not an elementary function was due to J. Liouville (1809 – 1882) and published during the late 1830s. A precise description of Liouville's proof is beyond the scope of a first year calculus course, but a relatively elementary and accessible account appears in the following expository paper:

> M. Rosenlicht, *Integration in finite terms*. American Mathematical Monthly **79** (1972), 963 – 972.

More general discussions (without proofs) of functions with nonelementary integrals appear in the following online sites:

http://mathworld.wolfram.com/ElementaryFunction.html

http://www.sosmath.com/calculus/integration/fant/fant.html

http://en.wikipedia.org/wiki/Nonelementary_integral

http://www.math.unt.edu/integration_bee/AwfulTruth.html

Many results on functions with nonelementary integrals were obtained during the next 100 years beginning with Liouville; the results of P. Chebyshev (1821 – 1894) on this topic were also particularly noteworthy and prove that functions like $\mathrm{sqrt}(1+x^4)$ and $1/\mathrm{sqrt}(1-x^4)$ do not have "nice" indefinite integrals, and more abstract and general results were obtained by 20th century mathematicians such as J. F. Ritt (1893 – 1951). About 40 years ago such questions were reconsidered from an algorithmic perspective, and the question of whether a function has an elementary integral was completely solved by R. H. Risch. The following online sites discuss these results without going into extensive detail:

http://mathworld.wolfram.com/RischAlgorithm.html

http://en.wikipedia.org/wiki/Risch_algorithm

http://www.freesoft.org/Classes/Risch2006/

Final remark. Although the integral of $\exp(-ax^2)$ is not an elementary function, one can write down a Maclaurin series expansion for this integral fairly easily starting with the usual power series expansion for the exponential function. Specifically, aside from the constant of integration this power series expansion only involves **odd** powers of x, and the coefficient of x^{2n+1} is given as follows:

$$\frac{(-1)^n a^n}{n!\,(2n+1)}$$

If $|x|$ is reasonably small, this series converges quite rapidly; in fact, by the basic results on alternating series, the error in approximating the integral by the Maclaurin polynomial of degree $2n+1$ has an absolute value less than or equal to the absolute value of the term of degree $2n+3$.