14.E. Calculus and problems in algebra

Ever since the time of the Greeks, if not earlier, mathematicians have realized that one branch of the subject is often extremely helpful in studying another. This is particularly apparent in the Greeks' systematic use of geometry to deal with irrational numbers and to solve equations that we now view as basically algebraic; for example, several books in Euclid's ***Elements*** include large amounts of material devoted to formulating algebra geometrically, and there are very lengthy treatments of issues related to irrational numbers. Conversely, the algebraic formulation of coordinate geometry in the 17^{th} century showed that algebraic methods made many basic geometrical concepts and problems much easier to handle. In the opposite direction, during the 18^{th} century several mathematicians began to notice ways in which the methods and results from calculus yielded fresh approaches to questions in algebra, which in some cases included strong new insights into basic questions in algebraic subjects like number theory and solutions of polynomial equations. In particular, input from calculus is fundamental to the proofs of Fermat's Last Theorem which were obtained near the end of the 20^{th} century. We shall simply discuss a few easily stated examples to illustrate the applications of calculus to questions which can be stated in purely algebraic terms.

Bernoulli numbers

One of the most basic results on power series is that the infinite series for the exponential function has the form

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots .$$

so that the coefficients for the various powers of x are given by a number – theoretic function; namely, $n! = 1 \cdot \ldots \cdot n$. The methods and results of calculus quickly led to the discovery of other important numerical sequences, and the ***Bernoulli numbers*** were an early example. As noted in the document

http://math.ucr.edu/~res/math153/bernoulli-numbers.pdf

this sequence first appeared in the early 17^{th} century, and many of its properties were studied independently by James Bernoulli and Seki; in particular, James Bernoulli noticed the formula

$$S_n(m) = \sum_{\nu=0}^{m-1} \nu^n = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} \frac{m^{k+1}}{k+1}$$

in which the (Bernoulli) numbers B_m are defined using the following power series expansion:

$$\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!} \ ,$$

Here is a table of the first few Bernoulli numbers:

$$B_1 = \frac{1}{6} \qquad B_{11} = \frac{854,513}{138}$$

$$B_2 = \frac{1}{30} \qquad B_{12} = \frac{236,364,091}{2,730}$$

$$B_3 = \frac{1}{42} \qquad B_{13} = \frac{8,553,103}{6}$$

$$B_4 = \frac{1}{30} \qquad B_{14} = \frac{23,749,461,029}{870}$$

$$B_5 = \frac{5}{66} \qquad B_{15} = \frac{8,615,841,276,005}{14,322}$$

$$B_6 = \frac{691}{2,730} \qquad B_{16} = \frac{7,709,321,041,217}{510}$$

$$B_7 = \frac{7}{6} \qquad B_{17} = \frac{2,577,687,858,367}{6}$$

$$B_8 = \frac{3,617}{510} \qquad B_{18} = \frac{26,315,271,553,053,477,373}{1,919,190}$$

$$B_9 = \frac{43,867}{798} \qquad B_{19} = \frac{2,929,993,913,841,559}{6}$$

$$B_{10} = \frac{174,611}{330} \qquad B_{20} = \frac{261,082,718,496,449,122,051}{13530}$$

(Source: http://www.makli.com/bernoulli-numbers-002031/)

Many properties of these numbers are described in the previously cited article, which is a cleaned up version of **http://numbers.computation.free.fr/Constants/constants.html**. For our purposes the most important point is the connection between these numbers and problems related to calculus.

Here is one easily described context in which Bernoulli numbers appear. We have already mentioned Euler's discovery that the sum of all terms $1/k^2$ equals $\pi^2/6$; in fact, if we let

$$\zeta(s) = \sum_{k=1}^{\infty} k^{-s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

then Euler's result fits into the following more general pattern:

$$\zeta(2n) = (-1)^{n+1} \frac{B_{2n}(2\pi)^{2n}}{2(2n)!}$$

Additional information about Bernoulli numbers is available from the following online references:

http://en.wikipedia.org/wiki/Bernoulli_number

http://mathworld.wolfram.com/BernoulliNumber.html

http://www.bernoulli.org/

Stirling's Formula. A related example deals with the factorial function $n!$; as noted above, the latter is the product of the first n positive integers (and by definition $0! = 1$). Simple computations show that this function grows very rapidly as n increases, and one natural question is to understand how the growth rate relates to that of other functions. During the first half of the 18th century one such estimate was obtained using ideas from calculus; this result is known as ***Stirling's Formula***:

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = 1,$$

This formula was discovered by J. Stirling (1667 – 1754) and (except for the constant factor involving the square root of 2π) independently by A. De Moivre (1692 – 1770); there is a discussion of the interaction between Sterling and De Moivre on pages 477 – 478 of Burton. Here are some references for further information:

http://en.wikipedia.org/wiki/Stirling%27s_approximation

http://www.math.uconn.edu/~kconrad/blurbs/analysis/stirling.pdf

We have already noted that Euler played an important role in applying the methods of calculus to number theory. In particular, he discovered the following attractive and fundamentally important infinite product identity (with an assumption that $s > 1$):

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}$$

Before discussing one important application of formulas like this, we shall first discuss an important application of methods from calculus to answer a basic question about finding roots of polynomials:

The Fundamental Theorem of Algebra. This is yet another important algebraic application of ideas from calculus which answers old questions about roots of polynomials; despite the result's name, ***all proofs of it use the continuity properties of the real numbers in some fashion and thus are intrinsically non – algebraic.*** On the other hand, the statement of the result is entirely algebraic:

> **EVERY** ***nonconstant*** *single – variable polynomial with complex coefficients has at least one complex root. Equivalently, all non – constant polynomials with complex coefficients are products of linear polynomials.*

As noted in **http://en.wikipedia.org/wiki/Fundamental_theorem_of_algebra** and page 548 of Burton, such a result had been conjectured in the 17th century, and there were numerous attempts to prove it during the 18th century, and the first indisputably complete proof was given by J. R. Argand (1768 – 1822) in 1806.

Primes in an arithmetic progression. Perhaps the most unexpected applications of calculus to algebraic involve the ***distribution of prime numbers***. Euclid's proof shows that there are infinitely many odd primes, and clearly one can split the odd primes into two subsets depending upon whether they leave a remainder of 1 or 3 when divided by 4. Some textbooks in elementary number theory or abstract algebra contain an exercise asking for a proof that the second of these classes is infinite, and it is natural to ask more generally about the number of primes in an arbitrary arithmetic progression. The following result, which is based upon methods from calculus and provides an optimally strong answer to the question, is due to J. P. G. Lejeune Dirichlet (generally abbreviated to ***Dirichlet***, 1805 –1859; the pronunciation is either *deer – i – clay* or *deer – i – shlay*):

> *Given two positive relatively prime integers* a *and* d, *the arithmetic progression* $a + nd$, *where* $n \geq 0$, *contains infinitely many primes.*

This and many other key connections between number theory and calculus can be traced back to Euler's infinite product identity which was described above. Further information on Dirichlet's result and related matters can be found on pages 559 – 560 of Burton and the online article **http://en.wikipedia.org/wiki/Dirichlet%27s_theorem_on_arithmetic_progressions**. As noted there, an "elementary" proof not using input from calculus was given much later by A. Selberg (1917 – 2007). In a related direction, during the first decade of the 21st century B. Green and T. Tao proved that the sequence of primes contains arbitrarily long arithmetic progressions (see **http://en.wikipedia.org/wiki/Green%E2%80%93Tao_theorem**). Many other questions involving the distribution of primes have been studied using methods from calculus, and the study of such problems has played an important role in mathematics during the past two centuries. Here are some references:

http://en.wikipedia.org/wiki/Prime_number_theorem

http://mathworld.wolfram.com/PrimeNumberTheorem.html

http://en.wikipedia.org/wiki/Riemann_hypothesis

Degrees of separation and unification in mathematics

In modern mathematics there are many contexts in which ideas from one branch of the subject often turn out to have important and far – reaching consequences for apparently unrelated problems. One can speculate whether or not this reflects the "six degrees of separation" theories which have been studied and written about in popular culture over the past few decades, but in any case they suggest a fundamental unity in mathematics which underlies the increasing number of different branches of the subject.

Here are some references which discuss evidence for and against the notion of "six degrees of separation." While it can be interesting or useful to speculate on this theory in particular examples, in each specific case it is also important to make sure that the supposed relations between two connected objects are actually ***significant***.

http://whatis.techtarget.com/definition/0,,sid9_gci932596,00.html

http://message.snopes.com/showthread.php?t=24972

http://en.wikipedia.org/wiki/Six_degrees_of_separation

http://en.wikipedia.org/wiki/Erd%C5%91s_number

http://www.oakland.edu/enp/