Math 153
Spring 2010
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SOLUTIONS TO EXERCISES FROM `math153exercises02.pdf`

As usual, "Burton" refers to the Seventh Edition of the course text by Burton (the page numbers for the Sixth Edition may be off slightly).

Problems from Burton, p. 103

3. By definition $t_n = (n^2+n)/2$. We need to show that $9\,t_n + 1$ is equal to t_m for some m. Let's see what happens if we substitute the expression for t_n into $9\,t_n + 1$.

$$9\left(\frac{n^2+n}{2}\right) + 1 \quad = \quad \frac{9n^2+9n+2}{2} \quad = \quad \frac{(3n+1)(3n+2)}{2} \quad = \quad t_{3n+1}$$

6. The number 1225 is equal to 35^2, and we follow the hint to write it as a triangular number t_n. According to the hint n is a root of the quadratic equation $n^2+n-2450 = 0$. The quadratic formula implies that the roots of the latter are equal to

$$\frac{-1 \pm \sqrt{1+4\cdot 2450}}{2} \quad = \quad \frac{-1 \pm 99}{2}$$

and hence 49 is the unique positive root. Therefore we see that $1225 = t_{49}$.■

Similarly, we check that $41616 = 204^2$ by some method, maybe using a calculator or table, maybe educated guessing, maybe the old algorithm for finding square roots that used to be taught in elementary schools (which might be discussed later in this course). We must then find the positive root of the quadratic equation $n^2+n-83232 = 0$, and this can be done using the quadratic formula:

$$n \quad = \quad \frac{-1 \pm \sqrt{1+4\cdot 83232}}{2} \quad = \quad \frac{-1 \pm \sqrt{332929}}{2} \quad = \quad \frac{-1 \pm 577}{2}$$

The positive root of this pair is 288, and therefore we have shown that $41616 = t_{288}$.■

11. (c) Follow the hint and write things out as

$$\sum_{k=1}^{n} k(k+1) \quad = \quad \sum_{k=1}^{n} k^2 + k \quad = \quad \sum_{k=1}^{n} k^2 \; + \; \sum_{k=1}^{n} k \quad =$$

$$\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \quad = \quad \frac{n(n+1)(n+2)}{3} \; .■$$

(d) Once again we use the hint:

$$\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} \quad = \quad \frac{1}{2}\cdot\sum_{k=1}^{n}\left(\frac{1}{2k-1} - \frac{1}{2k+1}\right) \quad =$$

$$\frac{1}{2} \cdot \left(1 \; - \; \frac{1}{2n+1} \right) \;\; = \;\; \frac{n}{2n+1}$$

This is an example of a telescoping series. The second piece of the k^{th} term cancels the first piece of the next term for every $k \leq n-1$.■

Problems from Burton, p. 116

2. Follow the hint, noting that if$x^2 + y^2 = z^2$ and $x + y + z = \frac{1}{2}\,xy$, then $(x-4)(y-4) = 8$. One could try to set up a system of quadratic equations, but in a simple case like this it is better to do things by hand. Since we are looking for integral solutions, this means that each of $x-4$ and $y-4$ must be 1, 2, 4 or 8. Thus the possible solutions to check are $(x,y) = (5,12), (6,8), (8,6) and (12,5)$. In the first and last cases we have $z = 13$, while in the middle two we have $z = 10$. Which of these choices for (x,y,z) also satisfy the equation $x + y + z = \frac{1}{2}\,xy$? Direct substitution shows that they all do.■

3. Following the hint, if n is odd we need to check that

$$n^2 \; + \; \left(\frac{1}{2}(n^2-1)\right)^2 \;\; = \;\; \left(\frac{1}{2}(n^2+1)\right)^2$$

and if n is even we need to check that

$$n^2 \; + \; \left(\frac{n^2}{4} - 1\right)^2 \;\; = \;\; \left(\frac{n^2}{4} + 1\right)^2 \; .$$

We do these as follows:

$$n^2 \; + \; \frac{n^4 - 2n^2 + 1}{4} \;\; = \;\; \frac{n^4 + 2n^2 + 1}{4} \;\; = \;\; \left(\frac{1}{2}(n^2+1)\right)^2$$

$$n^2 \; + \; \left(\frac{n^2}{4} - 1\right)^2 \;\; = \;\; n^2 \; + \; \frac{n^4}{16} \; - \; \frac{n^2}{2} \; + 1 \;\; =$$

$$\frac{n^4}{16} \; + \; \frac{n^2}{2} \; + 1 \;\; = \;\; \left(\frac{n^2}{4} + 1\right)^2 \; .$$

7. If we divide the equation $y_n^2 - 2\,x_n^2 = 1$ by $x_n 2$ we obtain

$$\left(\frac{y_n}{x_n}\right)^2 \; - \; 2 \;\; = \;\; \frac{1}{x_n^2} \; .$$

This means that $y_n \,/\, x_n$ will approach $\sqrt{2}$ in the limit if we know that $x_n \longrightarrow \infty$. However, by construction we know tht $x_n, y_n \geq 0$, and therefore the recursive relation $x_n = 3\,x_{n-1} + 2\,y_{n-1}$ implies $x_n \geq 3^{n-1} x_1 = 2 \cdot 3^{n-1}$. Thus x_n grows exponentially with n and as noted before this implies that the sequence of fractions $y_n \,/\, x_n$ has a limit which is equal to $\sqrt{2}$.■

18. We first show that h satisfies the harmonic mean equation if and only if it satisfies (b):

$$\frac{h-a}{b-h} \;\; = \;\; \frac{a}{b} \;\; \Longleftrightarrow \;\; b(h-a) \;\; = \;\; a(b-h) \;\; \Longleftrightarrow \;\; h(a+b) \;\; = \;\; 2\,ab \;\; \Longleftrightarrow$$

$$h = \frac{2\,ab}{a+b}$$

Next we show that h satisfies (a) if and only if it satisfies the harmonic mean equation:

$$\frac{1}{a} - \frac{1}{h} = \frac{1}{h} - \frac{1}{b} \iff \frac{h-a}{ha} = \frac{b-h}{hb} \iff$$

$$\frac{h-a}{b-h} = \frac{ha}{hb} = \frac{a}{b}$$

Problems from Burton, p. 127

4. Since the point (x, y) lies on the intersection of the two parabolas we know that $y^2 = 2\,ax$ and $x^2 = ay$. If we square the second equation we obtain $x^4 = a^2y^2 = a^2\,(2\,ax) = 2\,a^3x$. The real solutions of this equation are $x = 0$ and $x = \mathbf{cbrt}(2) \cdot a$. Since the intersection point has a positive first coordinate, the second solution is the one we want.■

Problems from Burton, p. 511

3. (a) The sum of all the proper divisions of p^n is

$$\sum_{k=0}^{n-1} p^k = \frac{p^n - 1}{p-1}$$

so if p^n were a perfect number then the right hand side would be equal to p^n. Multiplying both sides of such an equation by $(p-1)$ clears out the fractions and yields $p^n - 1 = (p-1) \cdot p^n$. Now the right hand side is at least p^n, so this inequality cannot hold and we have a contradiction. The problem arises because we assumed p^n was perfect, and therefore this cannot be true.■

(b) Following the hint, if p and q are odd primes then $p-1$ and $q-1$ are at least 2 so that

$$(p-1) \cdot (q-1) \geq 2 \cdot 2 = 4\ .$$

If we expand the left hand side we obtain

$$p\,q - p - q + 1 \geq 4$$

which shows that $pq > p + q + 1$. Therefore $p\,q$ cannot be a perfect number.■

6. Since n is perfect we know that

$$\sum_{d|n} d = 2n$$

and if we divide everything by n we obtain

$$\sum_{d|n} \frac{d}{n} = 2\ .$$

We now make a change of variables, setting $e = n/d$. As d runs through all divisors of n, the new variable e also runs through all divisors of n, so we may rewrite the second sum as

$$\sum_{e|n} \frac{1}{e} \quad = \quad 2\ .$$

11. To see that p cannot be part of an amicable pair, notice that it only has one proper divisor; namely, 1. Thus the sum of the proper divisors is 1, and since the sum of the proper divisors of 1 is 0 this means that p and 1 do not form an amicable pair.

Turning to p^2, since we know its proper divisors are 1 and p, the other number in the amicable pair would have to be $p+1$. Thus we need to show that the sum of the proper divisors of $p+1$ is not equal to p^2. Now the largest proper divisor is $\frac{1}{2}(p+1)$. This has two consequences. First, this is an upper bound on the size of any proper divisor. Second, there are at most $\frac{1}{2}(p+1)$ proper divisors of $p+1$. Thus the sum of the proper divisors is less than or equal to the upper bound on the number of proper divisors times the upper bound on the size of all such divisors. Upper bounds for both are given by $\frac{1}{2}(p+1)$, so a bound for the sum of the proper divisors is given by this number times itself, which is $\frac{1}{4}(p+1)^2$, If p^2 and $p+1$ were an amicable pair, this would mean that p^2 would be the sum of all the proper divisors of $p+1$ and consequently would be less than or equal to $\frac{1}{4}(p+1)^2$. However, if p is an odd prime then $p > \frac{1}{2}(p+1)$, and squaring this equation yields

$$p^2 > \tfrac{1}{4}(p+1)^2$$

which contradicts our previous conclusion. This means that p^2 cannot be part of an amicable pair.■

SOLUTIONS TO ADDITIONAL EXERCISES

0. In either case x and y differ by an odd number, so we have $x - y = 2k$ for some integer k. Thus we also have $x + y = (x - y) + 2y = 2k + 2y = 2(k + y)$ so their sum is also even. To finish off the argument use the identity $x^2 - y^2 = (x-y)(x+y) = 2k \cdot 2(k+y) = 4(k^2 + ky)$, which clearly shows that the left hand side is divisible by 4.■

1. (a) The first part is essentially worked out in one of the exercises from Burton.

(b) Use the same trick that we employed to show there are infinitely many Egyptian fraction expansions. Start with an arbitrary expansion

$$r \quad = \quad \frac{1}{q_1} \ + \ \cdots \ + \ \frac{1}{q_k}$$

where the terms are in decreasing order as usual, and consider the new expansion in which the final term is replaced using

$$\frac{1}{q_k} \quad = \quad \sum \frac{1}{dq_k}$$

where once again the terms on the right side appear in decreasing order. Putting these together, we obtain

$$r \quad = \quad \frac{1}{q_1} \ + \ \cdots \ + \ \frac{1}{q_{k-1}} \quad + \ \sum \frac{1}{dq_k}$$

and since $1/q_k$ was the smallest fraction in the first expansion, it follows that the terms in the new expansion are again in decreasing order. The smallest term in this expansion is $1/mq_k$, and its denominator is clearly divisible by m.■

2. Notice that we are adding dots along three of the five sides of the pentagon. Along one edge of the pentagon we are adding n dots, along the adjacent one we are adding another $n-1$ dots, and likewise along the remaining edge we are adding another $n-1$ dots. Thus the number P_n is $3n-2$ plus the previous number P_{n-1}.

One can verify the formula by induction, but here is another approach: The triangular numbers are given by

$$T_n \quad = \quad \sum_{k=1}^{n} k$$

and by the preceding observation the pentagonal numbers are given by

$$P_n \quad = \quad \sum_{k=1}^{n} 3n-2 \quad = \quad 3\cdot\sum_{k=1}^{n} k \quad - \quad \cdot\sum_{k=1}^{n} 2 \quad = \quad 3T_n \quad - \quad 2n \ .$$

If we now use the formula $T_n = \frac{1}{2}\cdot n(n+1)$ and simplify, we obtain the formula for P_n as stated in the exercise.

3. We shall refer to the picture repeatedly. The area of the region on the left bounded by the circle is $\pi = A + 2B + 4C$. Since the area of a circular sector with central angle θ and radius 1 is $\frac{1}{2}\theta$, it follows that $B + C = \pi/6$. Since the area bounded by an equilateral triangle with sides of length 1 is $\frac{1}{2}\sqrt{3}$, it follows that the latter equals B.

It follows that

$$C \quad = \quad \frac{\pi}{6} \quad - \quad \frac{\sqrt{3}}{2} \qquad \text{and} \qquad A \quad = \quad \pi \quad - \quad 2B \quad - \quad 4C \quad =$$

$$\pi \quad - \quad \sqrt{3} \quad - \quad \left(\frac{2\pi}{3} \quad - \quad 2\cdot\sqrt{3}\right) \quad = \quad \frac{\pi}{3} \quad + \quad \sqrt{3}\ .$$

This calculates out in decimal terms to 2.7792484..., which is about 88.5 per cent of the area bounded by the circle containing the given lune.∎