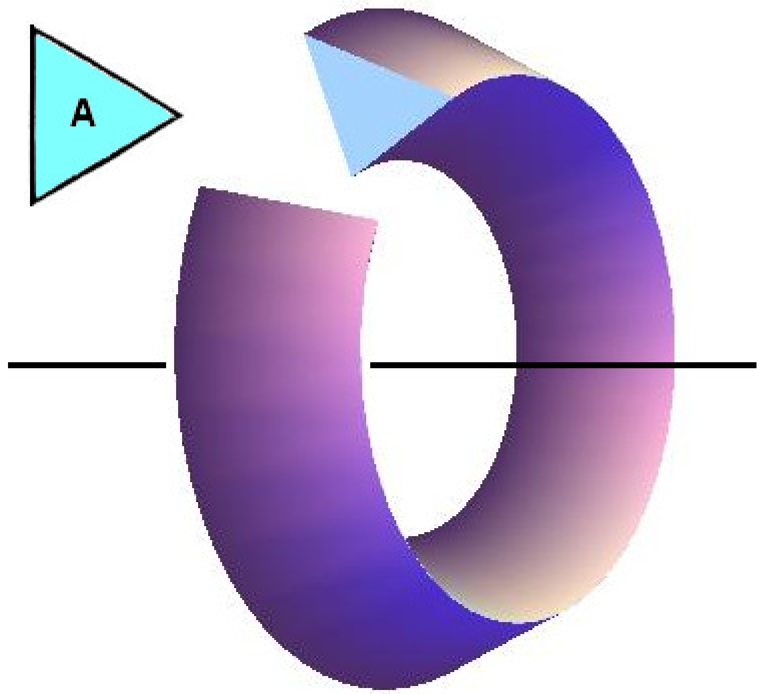
Another exercise using Pappus' Centroid Theorem

Let **A** be the region in the plane bounded by the equilateral triangle whose vertices are $(\mathbf{0},\ \boldsymbol{r}+\boldsymbol{a})$, $(\mathbf{0},\ \boldsymbol{r}-\boldsymbol{a})$, and $(\boldsymbol{a}\sqrt{\mathbf{3}},\ \boldsymbol{r})$, where $\boldsymbol{r} > \boldsymbol{a}$. Find the volume of the solid of revolution formed by rotating **A** about the $\boldsymbol{x}$ – axis.

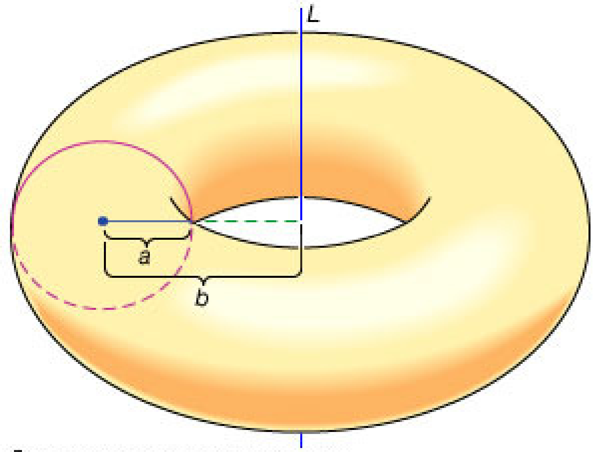
Here are drawings of the region **A** and the solid of revolution with a small piece removed.



Pappus' Centroid Theorem provides a very simple way of computing the volume with a minimum of effort. One piece of necessary input (say from ordinary geometry) is that the centroid of the equilateral lies on the line joining a vertex to the midpoint of the opposite side, so that it lies on the horizontal line $\boldsymbol{y} = \boldsymbol{r}$. We also need the formula for the area of the region bounded by an equilateral triangle whose edges have length $\mathbf{2}\boldsymbol{a}$, which is given by $\boldsymbol{a}^{\mathbf{2}}\sqrt{\mathbf{3}}$. By Pappus' Theorem the volume is the area of **A** times the circumference of the circle which is centered at the origin, is perpendicular to the $\boldsymbol{x}$ – axis, and passes through the centroid of **A** (and hence lies on the given horizontal line). Substituting these values into the general formula, we see that the volume is equal to

$$\left(\boldsymbol{a}^{\mathbf{2}}\sqrt{\mathbf{3}}\right)\cdot \mathbf{2}\boldsymbol{\pi}\boldsymbol{r} \;=\; \mathbf{2}\boldsymbol{\pi}\,\boldsymbol{a}^{\mathbf{2}}\boldsymbol{r}\sqrt{\mathbf{3}}\,.$$

On the next page there is a drawing corresponding to another volume problem which can be solved easily using Pappus' Centroid Theorem; namely, finding the volume bounded by a doughnut – shaped surface formed by rotating a circle of radius $\boldsymbol{a}$ about an axis whose distance from the circle's center is $\boldsymbol{b}$.



For this example the volume is equal to $\boldsymbol{2\pi^2 a^2 b}$**.**

COMPLEMENTARY PROBLEMS. Use the Pappus Centroid Theorem for surface areas to compute the surface areas of the two surfaces described above**.** In the first case it will probably be convenient to split the problem into three parts corresponding to the three "smooth" pieces of the surface**.**