Math 153
Spring 2012
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Preparation suggestions for the third examination

The third examination will be about 66 per cent problems and 34 per cent historical or short answer with extra credit possible on this portion of the exam. This exam will cover the material beginning with the final parts of the file `history07.pdf` (starting with Oresme's work) and continuing through the remaining files `history`n`.pdf`, where n runs from 8 through 14; the coverage also includes the corresponding files of exercises and solutions for these units. Some of the supplementary files especially worth reviewing are `history08a.pdf`, `history08b.pdf`, `history08c.pdf`, `history08d.pdf`, `history09a.pdf`, `history09b.pdf`, `history11a.pdf`, `history12a.pdf`, `history14a.pdf`, `history14b.pdf`, `history14c.pdf`, `history14d.pdf`, `history14e.pdf`, `history14f.pdf`, `cubic-example.pdf`, `fourier-uniform.pdf`, `impedance.pdf`, `impedance2.pdf`, `log-examples.pdf`, and `seriesexample.pdf`. The file `history02f.pdf` is also relevant to the period covered by the examination.

For the historical part, one main thing is to know the sequence of important developments and of important figures listed in the notes, basically in order and accurate to about a century or slightly less. There will also be some questions involving comparing the works of two or more mathematicians and their contributions to a major branch of the subject such as coordinate geometry or calculus. A review summary of the historical figures is included at the end of this document. No names from the period covered in the exam are on the first page of this summary, and there is only one (Zu Chogzhi, 429–501) on the second, but the whole list is printed for the sake of completeness.

For both the mathematical and parts, the problems in the exercises and the old examinations (see the subdirectory `aabOldExams` in the course directory) are good practice material. Also, a thorough understanding of background material like elementary algebra and geometry, precalculus, and first year calculus will be assumed, and problems drawing upon such knowledge are likely to be on the examination. Some aspects have been mentioned explicitly in the course notes, but a few others may appear. Here are a few areas worth studying as preparation:

1. The lectures mentioned the problems of (a) maximizing the area of a rectangular region bounded by a rectangle of fixed perimeter, (b) minimizing the perimeter of a rectangular region with fixed area. Still more maximization and minimization problems appear in the file `math153exercises12.pdf`.

2. There will probably be a problem related to the algebraic steps in the derivation of the cubic formula.

3. Something related to Cavalieri's Theorem may appear (possibly just knowing the satement of this result, possibly more).

4. Something related to the arithmetic of complex numbers (addition, subtraction, multiplication, division) may appear.

5. There may be something related to logarithms like the following: If $\log_{10} 3 \approx 0.4771$, how many digits are in the usual base 10 description of the integer 3^{81}? [Recall that $\log_{10} a^b = b \log_{10} a$ and if N is an integer then the number of digits in the base 10 expansion of N is equal to $1+[\log_{10} N]$, where $[x]$ denotes the greatest integer $\leq x$.]

6. There might be a question related to Kepler's wine barrel problem.

7. There might be a question involving evaluation of infinite series like

$$\sum_k \frac{n^d}{2^k}$$

(where d is a positive integer) using methods from calculus (see `seriesexample.pdf`).

Working maximization and minimization problems

This is just a review of some of the basic principles for finding the maximum and minimum of a continuous function $f(x)$ on an interval $a \leq x \leq b$. The maximum and minimum values are found amoung the *critical values* and *boundary values* $f(x)$ for which one of the following is true:

(1) We have $f'(x) = 0$.

(2) $f'(x)$ is not defined or is infinite.

(3) x is one of the endpoints a or b.

In practice, the hardest part of a maximization or minimization problem is often describing the function to maximize or minimize in terms of only a single variable. For example, in Kepler's wine barrel problem we know that the volume $V = \pi r^2 h$, where r is the radius of the cylindrical barrel and h is its height, and at first this looks like a function of both r and h. However, the condition on the stick length implies that $L^2 = r^2 + h^2$, so we can solve for one of these variables in terms of the other, and in particular we may write the volume as a function of the height h in this case. If we do so we obtain the following:

$$V(h) \quad = \quad \pi r^2 h \quad = \quad \pi(L^2 - h^2)h \quad = \quad \pi L^2 h \; - \; \pi h^3$$

If we set $V'(h) = 0$ and solve for h, we get Kepler's answer $h = L/\sqrt{3}$.

For the sorts of problems studied in this course, the maximum or minimum usually occurs at a value of x such that $f'(x) = 0$.

Historical summary

Starred names are from the periods covered on the first two exams

(624 BCE - 548 BCE) Thales* — First historic figure, results in geometry.

(580 BCE - 500 BCE) Pythagoras* — Early and influential figure in development of mathematics, basic number-theoretic questions and some geometry.

(520 BCE - 460 BCE) Panini* — Work on formal rules of grammar which foreshadowed 20$^{\text{th}}$ century research on computer languages.

(490 BCE - 430 BCE) Zeno* — Formulated paradoxes which had a major impact on the subject.

(470 BCE - 410 BCE) Hippocrates of Chios* — Computed areas, wrote early but lost books on mathematics.

(460 BCE - 400 BCE) Hippias* — Quadratrix or trisectrix curve, good for trisection and circle squaring.

(428 BCE - 348 BCE) Plato* — Influential ideas about how mathematics should be studied.

(417 BCE - 369 BCE) Theaetetus* — Proof that all integral square roots of nonsquares are irrational.

(408 BCE - 335 BCE) Eudoxus* — Proportion theory for irrationals, method of exhaustion to derive formulas.

(384 BCE - 322 BCE) Aristotle* — Influential work on logic and its role in mathematics.

(380 BCE - 320 BCE) Menaechmus* — Early work on conics, duplication of cube using intersecting parabolas.

(350 BCE - 290 BCE) Eudymus* — Lost writings on the the history of Greek mathematics.

(325 BCE - 265 BCE) Euclid* — Organized fundamental mathematical material in the *Elements*, including material on geometry, number theory and irrational quantities.

(300 BCE $\pm$ 2 centuries) Pingala* — Writings on language contained substantial mathematical information, including binary numeration, reference to Fibonacci sequence, results on combinatorial (counting) problems.

(287 BCE - 212 BCE) Archimedes* — Computations of areas and volumes, study of spiral curve, methods for expressing very large numbers.

(310 BCE - 230 BCE) Aristarchus* — Heliocentric universe, astronomical measurements, simple continued fractions.

(280 BCE - 220 BCE) Conon* — Associate of Archimedes also associated with the Archimedean spiral

(276 BCE - 197 BCE) Eratosthenes* — Prime number sieve, earth measurements.

(262 BCE - 190 BCE) Apollonius* — Extensive work on properties of conic sections, use of epicycles.

(240 BCE - 180 BCE) Diocles* — Focal properties of conics.

(190 BCE - 120 BCE) Hipparchus* — Early work on trigonometry, use of latter in astronomy, results in spherical geometry.

(80 BCE - 25 BCE) Vitruvius* — Applications of geometry to architectural design.

(10 AD - 75) Heron* — Area of triangle expressed in terms of sides.

(60 - 120) Nicomachus* — Special curves, nongeometric treatment of arithmetic.

(70 - 130) Menelaus* — Spherical geometry.

(85 - 165) Claudius Ptolemy* — Trigonometric computations, astronomy.

(200 - 284 conjecturally) Diophantus* — Algebraic equations over the integers and rational numbers, shorthand (syncopated) notation for expressing algebraic concepts.

(220 - 280) Liu Hui* — Commentary on the classic Chinese *Nine Chapters on the Mathematical Art*, which was probably written during the 1st century BCE, measurement results and techniques anticipating integral calculus.

(335 - 395) Theon* — Influential editing of the *Elements*, commentaries.

(370 - 418) Hypatia* — Daughter of Theon, lost commentaries and writings on numerous subjects.

(400 - 460 conjecturally) Sun Zi* — Influential mathematical manual, containing first known problem involving the Chinese Remainder Theorem.

(410 - 485) Proclus* — Commentaries on earlier Greek mathematics and its history.

(429 - 501) Zu Chongzhi — Discovery of a version of Cavalieri's Theorem.

(475 - 524) Boëthius* — Commentaries and summaries of Greek mathematics that were widely used for many centuries.

(476 - 550) Aryabhata* — Base ten numbering system mentioned in his work, introduction of trigonometric sine function, more extensive and accurate tables of trigonometric functions.

(480 - 540) Eutocius* — Commentaries publicizing the work of Archimedes.

(598 - 670) Brahmagupta* — Base ten numbering system explicit, free use of negative and irrational numbers, zero concept included, work on quadratic number theoretic equations over the integers, some shorthand notation employed.

(790 - 850) al-Khwarizmi* — Influential work on solving equations, mainly quadratics, beginning of algebra as a subject studied for its own sake.

(800 - 870) Mahavira* — Arithmetic manipulations with zero, clarification of earlier work in Indian mathematics.

(836 - 901) Thabit ibn Qurra* — Original contribution to theory of amicable number pairs, extensive work translating Greek texts to Arabic.

(850 - 930) Abu Kamil* — Further development of algebra.

(850 - 930) Al-Battani* — Work in computational trigonometry and trigonometric identities.

(940 - 998) Al-Kuhi* — Generalized version of the compass for constructing conics other than the circle.

(940 - 998) Abu'l-Wafa* — Highly improved trigonometric computations, discussion of the mathematical theory or repeating geometric designs.

(950 - 1009) Ibn Yunus* — Trigonometric computations and identities.

(953 - 1029) Al-Karaji/Al-Karkhi* — Introduction of higher positive integer exponents and negative exponents, manipulations of polynomials, recursive proofs of formulas that anticipate the modern concept of mathematical induction.

(965 - 1040) Al-Hazen* — Groundbreaking experimental and theoretical research on optics and related mathematical issues.

(1048 - 1122) Khayyam* — Graphical solutions of cubic equations using intersections of circles and other conics, foundations of Euclidean geometry.

(1114 - 1185) Bhaskara* — Extremely extensive and deep work on number theoretic questions including solutions to certain quadratic equations over the integers.

(1130 - 1180) Al-Samawal* — Further work on polynomials, recursive proofs, formulation of the identity $x^0 = 1$.

(1170 – 1250) Fibonacci (Leonardo of Pisa)* — Introduction of Hindu-Arabic numeration to nonacademics, work on number theory including Fibonacci sequence, problems involving sequences of perfect squares in an arithmetic progression, Pythagorean triples.

(approximately 1200) Al-Hassar — Horizontal bar symbol for fractions (around the same time as Fibonacci).

(1192 – 1279) Li Zhi* — Research on algebraic equations and number theory, with applications of algebra to geometric problems.

(1201 – 1274) al-Tusi, Nasireddin* — Early work on making trigonometry a subject in its own right, foundations of Euclidean geometry.

(1202 – 1261) Qin Jushao* — Wrote *Mathematical Treatise in Nine Sections* which summarizes much of Chinese work in mathematics at the time and breaks new ground.

(1220 – 1280) al-Maghribi* — Commentaries on the apocryphal Books XIV and XV of Euclid's *Elements*.

(1225 – 1260) Jordanus Nemoriarus* — Limited use of letters, results on perfect versus non-perfect numbers, relations between spherical and plane geometry (via stereographic projection), problems related to physics.

(1238 – 1298) Yang Hui* — Research on algebraic an number-theoretic questions, including magic squares.

(1260 – 1320) Zhu Shijie* — Algebraic summation formulas, solutions to some higher degree polynomial equations in several unknowns.

(1285 – 1349) Ockham — Formulation of the concept of a limit, principle of expressing things as simply as possible (Ockham's razor).

(1313 – 1373) Heytesbury — Mean speed principle for uniformly accelerated motion.

(1323 – 1382) Oresme — Summations of certain infinite series, early ideas on the graphical representation of functions.

(1350 – 1425) Madhava — Early figure in the Kerala School of mathematics, infinite series formulas for inverse tangent and π.

(1370 – 1460) Parameshvara — Early version of the Mean Value Theorem in calculus in the Kerala school.

(1377 – 1446) Brunelleschi — First specifically mathematical study of drawing in geometric perspective.

(1380 – 1450) al-Kashi* — Free use of decimal fractions and (infinite) decimal expansions, computation of π, Law of Cosines.

(1444 – 1544[*sic*]) Nilakantha Somayagi — Computation of π via infinite series in the Kerala school.

(1401 – 1464) Cusa, Nicholas of — Early mention of cycloid curve, other contributions including speculation on infinity.

(1404 – 1472) Alberti — First written treatment of geometric perspective theory.

(1412 – 1492) Francesca — Most mathematical treatment of perspective during this time period.

(1412 – 1486) Al-Qalasadi — Early versions of some modern notational conventions.

(1436 – 1476) Regiomontanus — Numerous translations of classical works, definitive account of trigonometry as a subject in its own right.

(1445 – 1500) Chuquet — Early versions of some modern notational conventions, "zillion" nomenclature for large numbers.

(1462 – 1498) Widman — First appearance of plus and minus signs.

(1465 – 1526) Pacioli — Comprehensive summary of mathematics at the time, published in print.

(1502 – 1578) Nunes — Mathematical theory of mapmaking.

(1512 – 1592) G. Mercator — Mathematical theory of mapmaking, important map projection with his name.

(1465 – 1526) Ferro — Discovery of the cubic formula.

(1470 – 1530) La Roche — Early printed mathematics book with good notation for powers and roots.

(1471 – 1528) Dürer — Research and writings on geometric perspective.

(1471 – 1559) Tunstall — First printed mathematics book in English.

(1492 – 1559) Riese — Authoritative and influential book on arithmetic and algebra.

(1499 – 1545) Rudolff — Introduction of the radical sign $\sqrt{\ }$.

(1500 – 1557) Tartaglia — Independent derivation of cubic formula, extension to other cases.

(1501 – 1576) Cardan — Major work on algebra including cubic and quartic formula, phenomena involving complex numbers.

(1502 – 1578) Nunes — Systematic study of mathematical problems in mapmaking.

(1510 – 1558) Recorde — Introduction of an early form of the equality sign.

(1512 – 1592) G. Mercator — Creation of useful and popular map projection.

(1522 – 1565) Ferrari — Quartic formula for roots of a 4^{th} degree polynomial.

(1526 – 1573) Bombelli — Use of complex numbers, clarification of cubic formula in the so-called irreducible case.

(1540 – 1603) Viète — Major advances in symbolic notation including the use of letters for known and unknown quantities, results in the theory of equations, new insights into the properties of trigonometric functions and their identities, influential ideas and results about using algebraic methods to study geometric questions.

(1546 – 1601) Tycho Brahe — Famous astronomer who extensively used precursors of logarithms to carry out computations.

(1548 – 1620) Stevin — Popularization of decimals throughout Europe, work on centers of gravity, hydrostatics.

(1550 – 1617) Napier — Invention of logarithms.

(1552 – 1632) Bürgi — Independent invention of logarithms, findings published later than Napier and Briggs.

(1560 – 1621) Harriot — Introduction of symbolism in his works (modern inequality signs first appear here, inserted by editors).

(1561 – 1615) Roomen — Formulation of challenging algebraic problem solved by Viète.

(1561 – 1630) Briggs — Continued Napier's work and published tables of common base 10 logarithms.

(1564 – 1642) Galileo — Important examples of curves arising from moving objects, crucial experimental discoveries in many areas of physics, Galilean paradox regarding infinite sets.

(1571 – 1630) Kepler — Laws of planetary motion, use of infinitesimals to find areas, Wine Barrel Problem in maxima and minima, semi-regular polyhedra, sphere packing problem.

(1574 – 1660) Oughtred — Invention of $\times$ for multiplication, invention of the slide rule.

(1577 – 1643) Guldin — Rediscovery of Pappus' Centroid Theorem.

(1584 – 1667) Saint-Vincent — Integral of $1/x$, refutation of Zeno's paradoxes using the concept of a convergent infinite series.

(1588 – 1648) Mersenne — Extensive correspondence with contemporary mathematicians, center of network for scientific exchange, study of Mersenne primes.

(1591 – 1626) Bacon, Francis — Major work on formulating the Scientific Method.

(1595 – 1632) Girard — Trigonometric notation, formula for area of a spherical triangle.

(1596 – 1650) Descartes — Refinements of Viète's symbolic notation including the use of x, y, z for unknowns, introduction of coordinate geometry in highly influential publication *Discours de la méthode*, but not including key features like rectangular coordinates or many of the standard formulas. The work on coordinate geometry was greatly influenced by classical Greek geometers such as Apollonius and Pappus and also by the work of Viète. Several fresh insights into questions about roots of polynomials.

(1598 – 1647) Cavalieri — Investigations of areas and volumes, Cavalieri's cross section principle(s), integration of positive integer powers x^n by geometric means.

(1601 – 1652) Beaune — Early exposition of Descartes' discoveries.

(1601 – 1665) Fermat — Important insights in number theory, coinventor of coordinate geometry (closer to the modern form than Descartes in many respects), preliminary work aimed at describing tangent lines and solving maximum and minimum problems. The work on coordinate geometry was greatly influenced by Apollonius in some respects and Viète in others.

(1602 – 1675) Roberval — Motion-based definition of tangents, numerous results on cycloids.

(1608 – 1647) Torricelli — Computations of integrals, results on cycloids, discovery of solid of revolution that is unbounded but has finite volume.

(1615 – 1660) Schooten — Extremely influential commentaries on Descartes' work.

(1616 – 1703) Wallis — Free use of nonintegral exponents, extensive integral computations including x^r where r is not necessarily a positive integer, major shift to algebraic techniques for evaluating such integrals.

(1620 – 1687) W. Brouncker — Standard infinite series for $\ln(1 + x)$ (independently with N. Mercator).

(1620 – 1687) N. Mercator — Standard infinite series for $\ln(1 + x)$ (independently with Brouncker).

(1622 – 1676) Rahn — First use of the standard division symbol $\div$.

(1623 – 1662) B. Pascal — Many important contributions, including properties of cycloids and integration of $\sin x$.

(1625 – 1672) De Witt — Contributor to the development of coordinate geometry.

(1628 – 1704) Hudde — Free use of letters to denote negative numbers, standard formulas for slopes of tangent lines to polynomial curves.

(1629 – 1695) Huygens — Numerous contributions, including solution of Galileo's isochrone problem, very wide range of scientific discoveries and results on related mathematical issues.

(1630 – 1677) Barrow — More refined definition of tangent line, realization that differentiation and integration are inverse processes, integrals of some basic trigonometric functions.

(1633 – 1660) Heuraet — Mathematical description of arc length and computations for some important examples.

(1638 – 1675) Gregory, James — Integration of certain trigonometric functions, familiar power series for the inverse tangent, first attempt to write a textbook on advances leading to calculus.

(1640 – 1718) La Hire — Work on solic analytic geometry and other aspects of geometry.

(1664 – 1739) Seki — Independent discovery of many results in calculus, number theory and matrix algebra, all done in Japan when that country was almost completely cut off from the rest of the world.

(1643 – 1727) Newton — Coinventor of calculus. Formulation of earlier work in more general terms, recognition of wide range of applications, material expressed in algebraic as opposed to geometric terms. Main period of discovery in 1660s, publication much later. Work strongly linked to his study of physical problems, particularly planetary motion. His main work on the latter, *Principia*, was highly mathematical. He obtained the standard binomial series expansion for $(1 + x)^r$, where r is real. Notation for calculus included *fluxion* for derivative, *fluent* for integral and $\dot{x}$ for the derivative. Infinitesimals were not strongly emphasized, but the use of infinite series to express functions was stressed. Priority was placed on differentiation. Newton's applications of calculus were extremely important influence in determining the subsequent development of mathematics for well over a century. Other results include the binomial series, codiscovery of an important numerical approximation technique (the Newton-Raphson Method), study of third degree curves and other algebraic problems, and results on recursively defined sequences.

(1646 – 1716) Leibniz — Coinventor of calculus. Formulation of earlier work in more general terms, recognition of wide range of applications, material expressed in algebraic as opposed to geometric Main period of discovery in 1670s, published in the next decade. Infinitesimals were strongly emphasized. The Leibniz notation, including dy/dx for derivative and $\int \, y\,dx$ for integral, became standard. Emphasis was on finding solutions that could be written in finite terms rather than infinite series. Priority was placed on integration. In other mathematical directions, Leibniz envisioned the use of algebraic methods in logic, determinants and systems of linear equations, and he systematically developed the binary numeration system (aspects of the latter had been considered by mathematicians intermittently for at least two millennia). Leibniz also made extremely important contributions to philosophy.

(1648 – 1715) Raphson — Codiscovery (with Newton) of popular numerical approximation method.

(1654 – 1705) Bernoulli, Jacob/Jacques/James — Continued work on calculus and differential equations as well as many other important contributions; numerous specific and general results on curves, maximum and minimum problems.

(1661 – 1704) de L'Hospital — Publication of influential calculus book with formula bearing his name (purchased from Johann Bernoulli).

(1667 – 1748) Bernoulli, Jean/Johann/John — Continued work on calculus and differential equations as well as many other important contributions; discovery of L'Hospital's Rule, measurement problems, transcendental functions.

(1667 – 1748) de Moivre — Polar form of complex numbers $re^{i\theta} = \cos\theta + i\,\sin\theta$, also other important work.

(1676 – 1754) Riccati — Differential equations, names for hyperbolic functions (sinh, cosh, *etc.*).

(1685 – 1753) Berkeley — Extremely influential critique of infinitesimals in calculus ("ghosts of departed quantities").

(1685 – 1731) Taylor — Publication of series expansion and approximation formulas bearing his name.

(1698 – 1746) Maclaurin — Publication of previously known power series expansion bearing his name, geometrical studies, lengthy response to Berkeley phrased in classical geometric terms.

(1700 – 1782) D. Bernoulli — Research on a wide range of physical problems and related mathematical issues, including trigonometric series.

(1707 – 1783) Euler — Extremely important contributions to many areas of mathematics, including number theory, mathematical notation infinite series, solid analytic geometry, and mathematical questions related to problems from physics.

(1713 – 1765) Clairaut — Development of solid analytic geometry, other contributions.

(1717 – 1783) d'Alembert — First suggestion of a concept of limit to circumvent logical problems with infinitesimals, various questions related to physics and the philosophy of science.

(1749 – 1827) Laplace — Mathematical questions related to a wide range of problems from physics.

(1765 – 1802) Ruffini — First effort to prove that no quintic (5^{th} degree) formula exists.

(1768 – 1830) Fourier — Fundamental studies of the trigonometric series named after him, mathematical issues related to heat conduction.

(1777 – 1855) Gauss — Extremely important contributions to many areas of mathematics.

(1781 – 1848) Bolzano — Wide ranging studies and analyses of foundational questions in calculus, including the Intermediate Value Theorem for continuous functions.

(1789 – 1857) Cauchy — Mathematical definition of limit in 1820 – nearly 150 years after the publication of Leibniz' work, conditions for convergence of sequences and series, essential features of the modern definition of derivative, uncoupling of differentiation and integration concepts from each other, also many other important contributions.

(1802 – 1831) Abel — Improved argument that radical formulas for roots of polynomials with degree ≥ 5 do not exist, insistence on a logically rigorous development of infinite series, other extremely important and far-reaching contributions over a very short lifetime.

(1805 – 1859) Dirichlet — Basic result on convergence of trigonometric series, definition of function close to the modern formulation, several other important contributions.

(1815 – 1897) Boole — Systematic introduction of algebraic methods into logic.

(1815 – 1897) Weierstrass — The modern $\varepsilon - \delta$ definition of a limit, convergence conditions for infinite series, also many other important contributions.

(1821 – 1881) Heine, Eduard — Maximum and Minimum Value Theorems for continuous functions.

(1826 – 1866) Riemann — Extremely important contributions to many areas of mathematics, including the standard definition of integrals in undergraduate textbooks.

(1831 – 1916) Dedekind — Mathematically rigorous description of the real number system, also many other important contributions.

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(1845 – 1918) Cantor — Theory of infinite sets (the logical foundation of modern mathematics), rigorous description of the real number system.

(1850 – 1891) Kovalevskaya — Conditions for convergence of series arising from problems in physics.

(1858 – 1932) Peano — Crucial advances on foundational questions, including a simple set of axioms for the nonnegative integers, examples of space-filling curves.

(1862 – 1943) Hilbert — Contributions to an extremely broad range of mathematical problems, extremely influential views on the theory and practice of mathematics.

(1870 – 1924) Koch — Important example of a fractal curve (Koch snowflake, bounded with infinite arc length but with many highly regular and symmetric features).

(1875 – 1941) Lebesgue — Definitive concept of integral in modern mathematics.

(1882 – 1885) E. Noether — Extremely influential changes in mathematicians' views of algebra, moving from solving equations to studying abstract systems which satisfy suitable axioms.

(1887 – 1920) Ramanujan — Extraordinarily original work on number theory.

(1906 – 1978) Gödel — Fundamental breakthrough results on the limits of logic for studying infinite mathematical systems, other results and constructions in the foundations of mathematics.

(1916 – 2001) Shannon — Importance of the binary numeration system for computer arithmetic.

(1918 – 1974) Robinson — Logically rigorous formulation of infinitesimals (non-standard analysis).

(1924 – 2010) Mandelbrot — Fractal curves (irregular bounded curves with infinite arc length, but with many interesting and useful properties).