Math 205A
Fall 2014

UPDATED GENERAL INFORMATION — OCTOBER 10, 2014

Assignments for Unit **II**

Working the exercises listed below is **strongly recommended.**

The following exercises are taken from Munkres:

- Munkres, Section 13: 3
- Munkres, Section 16: 1, 3
- Munkres, Section 17: 8, 19, 20
- Munkres, Section 18: 2 – 4, 9*c*, 11
- Munkres, Section 20: 3*b*

The following references are to the file file `gentopexercises2014.pdf` in the course directory.

- Additional exercises for Section II.1: 0, 3, 4
- Additional exercises for Section II.2: 0–7, 10, 13, 16–18
- Additional exercises for Section II.3: 1, 3, 8
- Additional exercises for Section II.4: 3, 4, 6, 10–12

Reading assignments from solutions to exercises

Another strong recommendation is to read through the solutions to the following problems in the files `math205Asolutions0*.pdf`, where $* = 1$ for Section II.1 and $* = 2$ for the remaining sections; some additional comments on the significance of these exercises are also given below.

- Munkres, Section 18: 6
- Additional exercises for Section II.1: 1, 5
- Additional exercises for Section II.2: 9, 15
- Additional exercises for Section II.3: 2, 5, 6, 9
- Additional exercises for Section II.4: 1, 2, 5, 7

Other reading assignments

- Read and understand the meaning of the first two results in Appendix C to `gentop-notes2014.pdf`, and understand how they generalize Example 1 on the first page of `homeomorphisms.pdf` (there is a drawing for the example on the third page of that document).
- Look at the construction in `flattening.pdf` for further examples of how a homeomorphism can bend, stretch and shrink subsets of the plane.

The following items involve topics from undergraduate courses; in each case there are connections to the material in Unit II. These are optional reading assignments, but they might be helpful in reinforcing or enhancing one's understanding of some concepts in Unit II.

- Look at Section V.2 of `linalgnotes.pdf`. The results of this section imply that for every hyperquadric in $\mathbb{R}^n$ (these are subsets defined by nontrivial quadratic polynomial equations in n variables) there is a homeomorphism from $\mathbb{R}^n$ to itself which sends the hyperquadric to one standard example in a relatively short finite list (*e.g*, in $\mathbb{R}^2$ the main examples are a circle or ellipse, a hyperbola, a parabola, or a pair of lines).
- There is an application of the results in the preceding item to a geometrical question about tangent lines in `elltangents.pdf`. The latter illustrates how one can use geometrical and topological transformations (in this case, affine transformations) to reduce the proof of a theorem about ellipses to a theorem about the standard unit circle.
- Look at the file `affine+measure.pdf` for some very simple comments on affine transformations which are area or volume preserving. More generally, the relations between homeomorphisms of $\mathbb{R}^n$ and measure theory have been studied extensively, but the methods and results are beyond the scope of this course.
- Read through the discussion of reflections — which are maps sending a point to its mirror image with respect to a line in $\mathbb{R}^2$— in `reflections.pdf`, which is a typical example of how one translates a geometrically describable transformation into explicit algebraic or analytic terms. It would also be worthwhile to derive a generalization of the results in this document to reflections about a plane in $\mathbb{R}^3$, or even more generally about a hyperplane in $\mathbb{R}^n$ (which is either defined by a nontrivial polynomial of degree 1 or as a subset of the form $x + V$ where $x \in \mathbb{R}^n$ and V is a subspace of dimension $n - 1$).

Comments

As we have noted in several other documents, students are assumed to have seen some versions of the main results in this unit, probably in undergraduate real variables courses and possibly in undergraduate point set topology courses. The main point here is to set things up in greater generality, and some of the exercises are meant to give mathematically significant examples of systems which satsify the abstract definitions. Exercise II.1.1 deals with an example called the **p-adic integers**, which arises in connection with problems in algebra and number theory. Other examples, which arise naturally in real analysis (upper and lower semicontinuity), are given in Additional Exercise II.1.5. The point of Additional Exercise II.2.9 is to emphasize that there are several alternative approaches to defining topological spaces beyond specifying their open subsets or their closed subsets (see also Additional Exercise II.1.6; this alternate characterization turns out to be particularly useful in practice). We shall use Additional Exercises II.2.15 in some subsequent discussions, and for this reason we have placed it on the reading list.

We now turn to the readings of solutions for Section II.3. Exercise 18.6 in Munkres describes a standard example of a highly discontinuous function from undergraduate real variables courses. Additional Exercise II.3.2 provides a powerful and useful tool for constructing new continuous functions out of old ones, and its applications are not limited to real analysis. Alternate characterizations of open and closed mappings, which resemble some alternate characterizations of continuous mappings, are established in Additional Exercise II.3.3. Both Additional Exercise II.3.5 and II.3.6 illustrate the difference between joint and separate continuity for functions of two variables; students have probably seen similar examples in multivariable calculus courses, so these are essentially review. Finally, Additional Exercise II.3.9 contains additional specific information on the upper and lower semicontinuity topologies.

Additional Exercise II.4.1 discusses a result which probably seems obvious (a Cartesian product of Cartesian products is a Cartesian product), but some reasoning is needed to justify this assumption, and the solution to the exercise gives an argument based upon the Universal Mapping Property for products in `characterizations.pdf`. This property also figures in the proofs of several results in Section II.4 including Proposition II.4.9, and Additional Exercise II.4.5 is essentially a corollary of that result. The purpose of Additional Exercise II.4.2 is to provide an example of a proof which almost completely avoids the Hausdorff Separation Property; spaces which satisfy the hypothesis of the exercise arise in other branches of mathematics, and particularly in algebraic geometry. Finally, Additional Exercise II.4.7 provides one more principle for comparing the different norms $|x|_p$ on $\mathbb{R}^n$ which were introduced in this unit; namely, there are homeomorphisms from $\mathbb{R}^n$ to itself such that the set of all points satisfying $|x|_\alpha = r$ is sent to the set of all points satisfying $|h(x)|_\beta = r$ for every $r \geq 0$. One consequence (noted in the exercise) is that a solid round disk of radius r in $\mathbb{R}^n$ is homeomorphic to a solid hypercube in $\mathbb{R}^n$ whose edges all have length $2r$; of course, we can then use the standard radial shrinking and stretching maps $x \to cx$ to say that every round solid n-dimensional disk is homeomorphic to every n-dimensional hypercube.