Math 205A
Fall 2014

UPDATED GENERAL INFORMATION — OCTOBER ??, 2014

Assignments for Unit **III**

Working the exercises listed below is **strongly recommended.**

The following exercises are taken from Munkres:

- Munkres, Section 23: 2–5, 9, 12* assuming Y is closed
- Munkres, Section 25: 1b, 9b, 10ab
- Munkres, Section 26: 3, 7, 8
- Munkres, Section 27: 2$abcd$

The following references are to the file `gentopexercises2014.pdf` in the course directory.

- Additional exercises for Section III.1: 1
- Additional exercises for Section III.3: 5
- Additional exercises for Section III.4: 1–3, 5–7
- Additional exercises for Section III.5: 1, 3, 5

Reading assignments from solutions to exercises

Another strong recommendation is to read through the solutions to the following problems in the files `math205Asolutions03.pdf`; some additional comments on the significance of these exercises are also given below.

- Munkres, Section 24: 10c
- Additional exercises for Section III.1: 1, 5
- Additional exercises for Section III.3: 9, 15
- Additional exercises for Section III.4: 4, 8
- Additional exercises for Section III.5: 2, 4

Comments

This unit begins with generalized versions of material from undergraduate real variables courses, but it quickly moves to concepts and results which generally do not appear in such courses. In many cases the statements or proofs are less straightforward than material in earlier course, much of this more sophisticated material material is fundamental to subsequent courses in the 205 sequence.

In most cases, the solutions assigned for reading are meant to illustrate certain points. For example, Additional Exercise III.1.5 is an example of what can happen if one weakens the hypotheses slightly (in this case, replacing a continuous function by a semicontinuous one); in this case, one ends up with a conclusion which is weaker but still nontrivial (namely, a lower semicontinuous function on a compact space takes a minimum value) but the proof is more than just a straightforward modification of the proof that continuous functions on compact spaces take maximum and minimum values. In contrast, Additional Exercise III.3.5 is an example of how one proves a plausible statement (in this case, density of the complement) for certain basic examples of subspaces.

Additional Exercise III.4.4 involves checking whether statements about topological spaces are true or false, developing the depth and breadth of knowledge needed to make educated guesses about truth or falsehood, and applying one's knowledge to either prove the truth of a statement or to come up with counterexamples. Similarly, Exercise 24.10*c* tests one's understanding of concepts with more complicated examples involving components, quasicomponents and arc components, and it also provides an example of working with nontrivial examples. The point of Additional Exercise III.4.8 is more sophisticated; one's intuition might suggest that the statement in the exercise is false (it is hard to visualize curves joining two points in $\mathbb{R}^2 - \mathbb{Q}^2$), but if one follows the argument in the solution it becomes necessary to concede that the statement is actually true. Additional Exercise III.5.2 is also meant as practice in finding counterexamples.

Finally, Additional Exercise III.5.4 is meant as practice following a moderately complicated argument which proves an extremely fundamental fact (for the circle, this can be shown as in Additional Exercise III.5.3, and for the 2-dimensional sphere one can establish this using spherical coordinates θ and ϕ where θ is between 0 and 2π and ϕ is between 0 and π).

Choosing a topology for an infinite product

Suppose that we are given an infinite product of topological spaces $\prod_\alpha X_\alpha$. In Section II.4 we gave one reason for choosing the product topology (whose basis is all products of open sets $\prod_\alpha U_\alpha$ such that $U_\alpha = X_\alpha$ for all but finitely many α) instead of the more easily defined box topology (whose basis is all products of open sets $\prod_\alpha U_\alpha$); namely, it is the smallest topology for which the coordinate projection mappings are continuous. As noted in Example 2 on page 117 of Munkres, one cannot conclude that a function into a product with the box topology is continuous if its coordinate projections are continuous.

There are two other respects in which the product topology behaves better than the box topology. Namely, the product of a family of compact spaces will always be compact in the product topology, and the product of a family of connected spaces will always be connected in the product topology. Both of these fail for the box topology. For compactness, one can take an infinite product of copies of the unit interval $[0, 1]$; if B denotes this space with the box topology and P denotes this space with the product topology, then the identity map from B to P is continuous and by Theorem 19.4 in Munkres (see p. 116) both spaces are Hausdorff. If B were compact, then the identity mapping would be a homeomorphism, contradicting the fact that the box topology is strictly larger than the product topology. For connectedness, one can take a countably infinite product of copies of $\mathbb{R}$ and proceed as follows: Let U be the set of all points $(x_1, x_2, \cdots) \in \mathbb{R}^\infty$ such that $\lim_{n\to\infty} x_n = 0$. For each point $(x_1, \cdots)$ in U the box open subset $\prod_n (x_n - \frac{1}{n}, x_n + \frac{1}{n})$ is contained in U and hence U is open in the box topology. On the other hand, for each point $(x_1, \cdots)$ in $X - U$ the box open subset $\prod_n (x_n - \frac{1}{n}, x_n + \frac{1}{n})$ is contained in $X - U$ and hence $X - U$ is also open in the box topology. Since U and $X - U$ are both nonempty, it follows that X is not connected.

An additional reference

The following book is another reference for the material discussed in the subheading *Strong arcwise connectedness* in Section III.5:

S.B. Nadler. Continuum theory. An introduction. Monographs and Textbooks in Pure and Applied Mathematics Vol. 158. *Marcel Dekker, New York*, 1992.