Math 205A
Fall 2014

UPDATED GENERAL INFORMATION — OCTOBER ??, 2014

Assignments for Unit **V**

Working the exercises listed below is **strongly recommended.**

The following references are taken from Munkres:

- Munkres, Section 22: 2, 4 (the solution to 22.2 is given below)

The following references are to the file `gentopexercises2014.pdf` in the course directory.

- Additional exercises for Section V.1: 0 ,7, 9
- Additional exercises for Section V.2: 1, 2, 5

Reading assignments from solutions to exercises

Another strong recommendation is to read through the solutions to the following problems in the files `math205Asolutions03.pdf`; some additional comments on the significance of these exercises are also given below.

- Additional exercises for Section V.1: 1 – 3

Solution to Munkres, Exercise 22.2

In both parts of the exercise one has the same given data; namely, a continuous mapping $f : X \to Y$ and a *cross section* $\sigma : Y \to X$ which is continuous and has the property that $\sigma \circ f = \mathrm{id}_Y$. Furthermore, the object in each part is to show that f is a quotient map; equivalently, we need to show that f is onto and V is open in Y if (and only if) $f^{-1}[V]$ is open in X. The "only if" implication follows by continuity of f, and f is onto because $y \in Y$ implies that $y = \sigma(f(y))$.

To complete the proof of the statement(s) in the exercise, suppose that $V \subset Y$ is such that $f^{-1}[V]$ is open in X. Then we have

$$V \quad = \quad \sigma^{-1}\left[\, f^{-1}[V]\,\right]$$

because $\sigma \circ f = \mathrm{id}_Y$, and since σ is continuous it follows that the set on the right hand side, which is an inverse image of an open subset, must be open.■

Remarks. 1. There are examples of retractions which are closed but not open, open but not closed, both open and closed, and neither open nor closed.

2. If X and Y are Hausdorff and f and σ are given as above, then $\sigma[Y]$ is closed because it is the set of all points $x \in X$ such that $\sigma \circ f(x) = x$. More generally, we can conclude that σ is a closed mapping, for if $E \subset Y$ is closed then $\sigma[E] = \sigma[Y] \cap f^{-1}[E]$. Finally, it is easy to check that σ is 1–1 ($\sigma(y_1) = \sigma(y_2) \;\; \Rightarrow \;\; y_1 = f\sigma(y_1) = f\sigma(y_2) = y_2$), so it follows that σ maps Y homeomorphically to $\sigma(Y)$ if X and Y are Hausdorff.

Comments

The remarks at the beginning of Unit V explain why quotients of topological spaces are important at more advanced levels, and such constructions will arise repeatedly in each of the three courses in the205 sequence. Disjoint unions have received limited coverage in texts because everything is fairly straightforward (in particular, they never were a significant research topic because of this); however, they are needed in numerous contexts, and one reason for discussing them so extensively in Section V.2 is for reference purposes.

Here are some comments on the solutions assigned for reading. Additional Exercise V.1.3 treats an example which will appear in the second part of the course and in subsequent courses from this sequence, and Additional Exercise V.1.1 – 2 give results needed in the solution to Additional Exercise V.1.3. Other properties of the example in this exercise are discussed in `rpn-in-rk.pdf`. Still further sources of information about this and related examples are the undergraduate level notes

`http://math.ucr.edu/~res/progeom/pg-all.pdf`

and other files in the directory `http://math.ucr.edu/~res/progeom`.

More supplementary material for Unit **V**

A file, `scissors+paste.pdf`, has been added to the course directory in order to provide further details related to the Scissors and Paste Problem discussed on pages 87 – 88 of `gentop-notes2014.pdf`. The discussion uses concepts from the subheading, *Disjoint unions of families of sets*, on pages 89 – 90 of the same document.