Math 205A
Fall 2014

UPDATED GENERAL INFORMATION — OCTOBER ??, 2014

Assignments for Unit **VI**

Working the exercises listed below is **strongly recommended.**

The following exercises are taken from Munkres:

- Munkres, Section 30: 9 (first part only), 10, 13, 14
- Munkres, Section 33: $2a$ (for metric spaces), $6a$, 8
- Munkres, Section 38: 2Λ, 3 (just give a necessary condition on the topology of the space)
- Munkres, Section 40: 2

The following references are to the file `gentopexercises2014.pdf` in the course directory.

- Additional exercises for Section VI.1: 1, 3, 4
- Additional exercises for Section VI.3: 1, 3, 4
- Additional exercises for Section VI.4: 1, 2, 5, 7, 9
- Additional exercises for Section VI.5: 3

Reading assignments from solutions to exercises

Another strong recommendation is to read through the solutions to the following problems in the files `math205Asolutions05.pdf`; some additional comments on the significance of these exercises are also given below.

- Munkres, Section 33: 6
- Additional exercises for Section VI.1: 2
- Additional exercises for Section VI.2: 1
- Additional exercises for Section VI.5: 1

The statement and proof of Corollary VI.5.9A (on page 2 of this document) should also be read and understood.

Comments

This unit deals with more advanced properties of topological spaces. In Section VI.1 several important countability properties for subspaces of $\mathbb{R}^n$ are studied abstractly, and one of the main results in Section VI.2 uses these ideas to prove important properties of compact metric spaces.

The material on separation properties and metrization theorems was particularly important when there was great interest in finding necessary and sufficient conditions for a topological space to be metrizable, but currently most topics in Sections VI.3 and VI.5 are less central than the material in the other sections of the unit. The main things to take away from Sections VI.3 and VI.5 are the statements of the basic definitions, the proof that compact $\mathbf{T_2}$ spaces are $\mathbf{T_4}$, the fact that all the separation properties are true in metric spaces, and the statements (but not the proofs) of the basic metrization theorems. The document `embeddings.pdf` contains additional material related to these sections.

Significance of metrization theorems. Frequently it is useful to have purely topological criteria for knowing that a topological space is metrizable, for metrizable spaces have many good properties (first countability, separation properties, every closed subset is a countable intersection of open subsets, *etc.*). However, the usefulness of such a result depends on how easy it is to check that the criteria for metrization are satisfied. The Urysohn Metrization Theorem, which gives necessary and sufficient conditions for metrizability of a second countable space, is particulary good in this respect for at least two reasons: The conditions in the theorem is fairly simple to state and easy to check, and many important examples of topological spaces are second countable.

One consequence of the results in this unit is the following purely topological characterization of compact metrizable spaces. The proof brings together many of the main ideas in Unit VI.

COROLLARY VI.5.9A. *A compact topological space X is metrizable if and only if it is $\mathbf{T}_2$ and second countable.*

Proof. ($\Rightarrow$) If X is metrizable it is $\mathbf{T}_2$, and by Proposition VI.1.6 we know that a compact metric space is second countable.

($\Leftarrow$) If X is compact and $\mathbf{T}_2$, then by Theorem VI.3.3 we know that X is also $\mathbf{T}_3$ (in fact, it is also $\mathbf{T}_4$). Since X is also second countable, the Urysohn Metrization Theorem (VI.5.9) implies that X is metrizable.■

In contrast, the more general metrization theorems for arbitrary topological spaces have proven to be less useful, largely because the criteria are generally difficult to check for many examples.

Reading assignments. As usual, the solutions assigned for reading are meant to illustrate certain points. Exercise 33.6 in Munkres has appeared on qualifying examinations in the past, and the result itself illustrates how the methods of the course can be applied to verify a general statement which may seem to be "intuitively obvious" and can be checked directly for many basic examples. Additional Exercise VI.1.2 is meant to illustrate the relative strength of the second countability property for topological spaces that are not homeomorphic to metric spaces; in particular, this example shows that a result which is true for separable metric spaces is not necessarily true for more general topological spaces. The conclusion for Additional Exercise VI.2.1 might be needed in 205C, but in any case it plays an important role in topics closely related to that course. Finally, Additional Exercise VI.5.1 analyzes spaces of continuous functions on open subsets of $\mathbb{R}^n$, in which the natural topology involves uniform convergence on compact subsets; this topological structure plays an important role in many studies of function families on noncompact spaces.

Exercises and readings on topological groups

Topological groups are discussed in Appendix A to `gentop-notes.pdf`. Although this topic will not be covered in the present course, it plays important roles in many (arguably, nearly all) branches of pure mathematics, and it is extremely worthwhile to learn at least a little about this subject. Specific reading recommendations include Appendix A up to, but not including, the subheading *Analysis* on page 124. Also suggested are working Supplementary Exercise 1 at the end of Section 22 in Munkres, reading the solutions to Supplementary Exercises 5 and 6 from the same set, working Exercise 31.8 in Munkres and Additional Exercise A.0, and reading the solutions to Additional Exercises A.1 – A.4.

Supplementary Exercise 5 looks at the topological structure on the cosets associated to a subgroup of a topological group, Supplementary Exercise 6 is an example, and Additional Exercises A.1 – A.4 describe important examples of compact Hausdorff topological groups of invertible matrices.

Paracompactness

Although the definition of a paracompact space is relatively complicated, the underlying concept turns out to be extremely important in many areas of geometry, topology and analysis (compare the comments on page 252 of Munkres). One reason for this is the existence of families of functions called partitions of unity, which often allow one to construct a continuous function with certain good properties out of functions which are *a priori* only known to exist locally. The following example, which is related to Theorem 41.8 in Munkres, illustrates the sorts of things which can be done:

PROPOSITION. *Let X be a metric space, and let W be an open neighborhood of $X \times \{0\}$ in $X \times \mathbb{R}$. Then there is a continuous function $\delta : X \to (0, \infty)$ such that the open set*

$$\{ \ (x,t) \ \in \ X \times \mathbb{R} \mid |t| \quad < \quad \delta(x) \ \}$$

is contained in W.

In this example, near each $x \in X$ one has a constant ε_x and an open neighborhood W of $(x, 0)$ such that $W \times (-\varepsilon, \varepsilon) \subset U$, so locally one has a function δ with the right properties (in fact, a constant function). The idea of paracompactness is that it gives a method for piecing together these functions which are good locally into a function which is good globally.

Proof. This is included mainly for the sake of completeness, and it is not part of the course material to be covered on examinations.

For each $x \in X$ there is an open neighborhood V_x of x and a positive number $\varepsilon(x)$ such that $V_x \times N_{\varepsilon(x)}(0) \subset W$. By A. H. Stone's Theorem (Munkres, Theorem 41.4) the space X is paracompact, and hence the open covering $\{V_x \mid x \in X\}$ has an open locally finite refinement $\mathcal{U} = \{U_\alpha\}$; by construction we can find positive constants $\varepsilon(\alpha)$ such that $V_\alpha \times N_{\varepsilon(\alpha)}(0) \subset W$.

Let $\{\phi_\alpha\}$ be a partition of unity dominated by $\mathcal{U}$; such a family of functions exists by Munkres, Theorem 41.7. Next, let

$$\delta(x) \quad = \quad \sum_\alpha \ \varepsilon(\alpha) \cdot \phi_\alpha(x) \ .$$

Formally this is an infinite sum, but by the local finteness of $\mathcal{U}$ each point $x \in X$ has a neighborhood in which only finitely many summands are nonzero, so the formula defines a continuous function.

Furthermore, since $\sum_\alpha \phi_\alpha = 1$ and each ϕ_α takes nonnegative values everywhere, it follows that for each x there is some β such that $\phi_\beta(x) > 0$, and from this it follows that $\delta(x) > 0$ for all $x \in X$.

Finally, we claim that if $|t| < \delta(x)$ then $(x, t) \in W$. Given $x \in X$, let $\alpha_1, \cdots, \alpha_k$ be such that $x \in U_\alpha$ only if $\alpha = \alpha_j$ for some j. If α^* is chosen such that $\varepsilon(\alpha^*)$ is the largest of the positive constants $\varepsilon(\alpha_1), \cdots, \varepsilon(\alpha_k)$, then $\delta(x) < \varepsilon(\alpha^*)$ and hence $|t| < \delta(x)$ implies that $(x, t) \in U_{\alpha^*} \times N_{\varepsilon(\alpha^*)}(0) \subset W$. Therefore $|t| < \delta(x)$ implies $(x, t) \in W$, as required.■

In some cases it is even possible to construct partitions of unity with certain additional properties; for example, if X is an open subset of $\mathbb{R}^n$ it is possible to construct the functions ϕ_α so that they have continuous partial derivatives of all orders. Such objects play an important role in parts of 205C.