Math 205A
Fall 2014

UPDATED GENERAL INFORMATION — OCTOBER 24, 2014

The first in-class examination (continued)

Here are partial solutions to the sample questions:

1. (i) A base for the product topology is given by all products $U \times V$ where U is open in X and V is open in Y. Since X and Y are indiscrete, the only nonempty product of this form is $X \times Y$, so the product topology is generated by this single set and hence is equal to $\{\emptyset, X \times Y\}$, which is the indiscrete topology.■

(ii) If X and Y are both finite, then the discrete and finitary topologies are the same, and since $X \times Y$ is finite in this case it follows that the product topology is the discrete topology, which is equal to the finitary topology by the finiteness of $X \times Y$.

On the other hand, suppose that X is infinite and Y consists of more than one point. Then if $y \in Y$ the set $X \times \{y\}$ is closed in the product topology but not in the finitary topology (it is an infinite proper subset!). Similarly, if X consists or more than one point and Y is infinite, then the finitary topology and the product topology are not the same.

To summarize, the product topology equals the finitary topology on $X \times Y$ if and only if either both X and Y are finite or one of $|X|$ and $|Y|$ is equal to 0 or 1.■

2. The product space $X \times X$ is compact, and the distance function $d : X \times X \to \mathbb{R}$ is continuous. Since the continuous image of a compact set is compact, it follows that the image of d is closed and bounded, so that this image has some maximal element. This maximal element has the form $d(a, b)$ for some $a, b \in X$.■

3. (i) FALSE. There are many examples, and one simple choice is $(0, 1) \cup (2, 3) \subset \mathbb{R}$.■

(ii) TRUE. If U is open in X and $y \in U$, then y has a neighborhood base of connected open subsets in U if it has such a neighborhood base in X.■

(iii) TRUE. If $A \subset (0, 1)$, then $(0, 1) \times (0, 1) \cup A \times \{0\}$ is arcwise connected. The cardinality of this family equals the cardinality of the family of all subsets in $\mathbb{R}^2$, and we know this is also the cardinality of the family of all connected subsets in $\mathbb{R}^2$.■

(iv) TRUE. If A is closed and C is a connected component of A, then C is closed in A; since every closed subset of A is closed in X (becasue A is closed in X), it follows that C is closed in X.

(v) FALSE. Let $B \subset \mathbb{R}^2$ be the graph of $\sin(1/x)$ where $x > 0$, and let A be the closure of B. Then B is an arc component of A but it is not closed in X.■

4. (i) It suffices to show that f_0 maps each equivalence class to a single point, and since there is only one equivalence class with more than one point the only nontrivial case to consider is the equivalence class $X \times \{0\}$. Since $f_0(x, 0) = (0, 0)$, the condition in the first sentence is satisfied.■

(ii) The quotient space $Cone(X)$ is compact because it is the continuous image of a compact space, and therefore by a result from Section III.1 the map f will define a homeomorphism onto its image in $\mathbb{R}^{n+1}$ provided it is 1–1.

If $f[x,t] = f[x',t']$ then $f_0(x,t) = f_0(x',t')$; since the second coordinate of $f(y,s)$ is equal to s it follows that $s = s'$. There are now two cases depending upon whether or not this coordinate is zero or positive. Since $[x,0] = [x',0]$ for all $x, x' \in X$, there is nothing to prove if the coordinate is zero because there is only one point in the cone with this coordinate. So suppose now that the first coordinate is positive. Then $t = t'$ and hence we have $f(x,t) = f(x',t)$. This can be rewritten as $(tx,t) = (tx',t)$, and since $t \neq 0$ it follows that $x = x'$.■

5. (*i*) We have $p(0,0) = 0$ and $p(1,1) = 1$. Since $[0,1] \times [0,1]$ is connected, it follows that for each $t \in (0,1)$ we can find $(A,B) \in [0,1] \times [0,1]$ such that $p(A,B) = t$. In particular, this is true for $1/\sqrt{2} \approx 0.707$.

(*ii*) If f is identically zero there is nothing to prove. Suppose that there is some (x_0, y_0) such that $f(x_0, y_0) > 0$. The limit condition implies that there is some $r > 0$ such that $x^2 + y^2 > r^2$ implies that $f(x,y) < \frac{1}{2}r$. Since f is continuous and the closed disk of radius r is compact, there is some (x_1, y_1) such that $f(x_1, y_1)$ is a maximum value on this disk, and this maximum value is at least r. Since $0 \leq f \leq \frac{1}{2}$ off this disk, it follows that $f(x_1, y_1)$ is the maximum value for all $(x,y) \in \mathbb{R}^2$.

For the final assertion in the exercise, observe that

$$f(x,y) \quad = \quad \frac{1}{1 + x^2 + y^2}$$

satisfies the conditions of the exercise and is positive everywhere; for suitable choices of x and y this function assumes every real value in the half open interval $(0,1]$, and therefore the function does not have a minimum value.■