Math 205A
Fall 2014

UPDATED GENERAL INFORMATION — NOVEMBER 17, 2014

The second in-class examination

The second in-class examination, which will take place on **Monday, November 24**, will cover the material from Unit VI in `gentop-notes.pdf` which is part of the course (more details are given below) as well as Unit VII and the first two sections of Unit VIII in `fundgp-notes.pdf`, with exceptions as noted below. Regarding the material on category theory (Section VII.0 of the course exercises and related files as indicated), there will be no problems on this exam which deal explicitly with category-theoretic notions.

The problems on the exam will be stating definitions and results in the course notes, giving proofs of results in the notes which are at the easy or moderately complicated level, of questions which are similar to the easy and moderately challenging exercises which were strongly recommended in earlier postings.

As usual in mathematics, concepts and results from earlier portions of the couse are likely to be needed for the statements of definitions and results, and also for the arguments proving results or solving exercises.

Here are a few sample questions to consider. Some might be more demanding than the problems which will appear on the exam but not dramatically so, and similarly some may require more time than the problems on the exam.

1. Let X and Y be metric spaces, let $f : X \to Y$ be continuous, and assume that X is second countable. Prove that $f[X]$ is also second countable. [*Hint:* Why does it suffice to prove the analogous statement when second countability is replaced by the Lindelöf Property?]

2. Let X be a $\mathbf{T}_3$ space, Let $K \subset X$ be a compact, and let $E \subset X$ be closed in X such that $E \cap K = \emptyset$. Prove that there are disjoint open subsets U and V in X such that $K \subset U$ and $E \subset V$.

3. For each of the statements below, indicate whether it is true or false and give brief reasons for your answer.

(i) If X is a connected locally compact $\mathbf{T}_2$ space, then X is homeomorphic to a subset of a connected compact $\mathbf{T}_2$ space.

(ii) If X is simply connected (arcwise connected and trivial fundamental group) and $f : X \to Y$ is continuous, then $f[X]$ is also simply connected.

(iii) If A is a compact subset of $\mathbb{R}^n$ and $a \in A$, then there is a compact subset $K \subset \mathbb{R}^n$ such that the inclusion $(A, a) \to (K, a)$ induces the trivial homomorphism $\pi_1(A, a) \to \pi_1(K, a)$.

4. Let $\{K_\alpha \mid \alpha \in A\}$ be a family of convex subsets in $\mathbb{R}^n$ such that there is some point $p \in \cap_\alpha K_\alpha$. Prove that $\cup_\alpha K_\alpha$ is contractible.

5. Let (X, x) be an arcwise connected space with basepoint, and assume that $\pi_1(X, x)$ is a finitely generated abelian group. Prove that $X \times S^1$ is not homeomorphic to X.

Included and excluded material

Here is a listing of some sorts of things that students are not responsible for on this exam, complemented by some strong recommendations on material to understand as well as possible.

It is important to understand the content of Section VI.1 as well as possible, and the same applies to Section VI.2, except that it is not necessary to have an active understanding of the proof that compactness and sequential compactness are equivalent for metric spaces. In Section VI.3, the definitions of the listed separation axioms should be known (along with the concepts of regularity and normality), but the material on non-Hausdorff topologies will not be covered in the exam. It is also important to understand all the content of Section IV.4 up to the statement of Theorem 10. To conclude the discussion of Unit VI, it is enough to know the statements of Proposition 8 and Theorem 9.

Regarding the material on homotopy and fundamental groups, it is important to understand the contents of Sections VII.1 and VII.3; the results of Section VII.2 will not be covered on the exam. In Section VII.4, the material up to the subheading, *A more complicated example*, should be understood thoroughly, and a passive understanding of the material in latter subheading is a good indication of how well one has mastered the topics in this unit. Nothing from `secVII4-addendum.pdf` will appear on the exam. The constructions and statements of results in Section VIII.1 should all be well understood, and a passive understanding of the proofs of the results is a good indication of how well one knows the material in this section. The statements of the first 6 results in Section VIII.2 are likely to appear in some form on the exam, and likewise for Theorem 11 in `secVIII2-addendum.pdf`.

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SOLUTIONS

1. Follow the hint. Since second countability, separability and the Lindelöf Propertyare equivalent for metrizable spaces, it suffices to show that if $f : X \to Y$ is continuous and X has the Lindelöf Property, then so does $f[X]$.

One can prove this by the same sort of argument employed to show that the continuous image of a compact space is compact. Let $\mathcal{U}$ be an open covering of $f[X]$, and for each U_α in U write $U_\alpha = W_\alpha \cap f[X]$ where W_α is open in Y. Then the sets $V_\alpha = f^{-1}[W_\alpha]$ form an open covering of X, and since X has the Lindelöf Property there is a countable subcovering

$$\{ f^{-1}[W_1], f^{-1}[W_2], \cdots \} .$$

We claim that $U_j = f\left[f^{-1}[W_j]\right]$ for each j; this is true because for every subset $B \subset Y$ we have

$$B \cap f[X] = f\left[f^{-1}[B]\right]$$

and if $B \subset f[X]$ (for example, if $B = U_j$) then B is equal to $f\left[f^{-1}[B]\right]$. Therefore, as in the proof for compactness we can conclude that $\cup_j U_j = f[X]$, so that the family $\{U_1, U_1, \cdots\}$ is a countable subcovering of $f[X]$.■

2. For each $x \in K$ we can find open subsets U_x and V_x which are disjoint and such that $x \in U_x$ and $E \subset V_x$. The sets U_x form an open covering of K, so take a finite subcovering $\{U_j\}$, where U_j is associated to x_j. If V_j is the open neighborhood of E which is paired to U_j, let $U = \cup_j U_j$ and $V = \cap_j V_j$. Then $K \subset U$, $E \subset V$, and

$$U \cap V = \left(\bigcup_j U_j\right) \cap V = \bigcup_j (U_j \cap V) \subset \bigcup_j (U_j \cap V_j)$$

where all subsets in the right hand expression are empty by hypothesis. Therefore U and V are disjoint open subsets containing K and U respectively.■

3. (i) TRUE. If X is compact, then we can take X itself to be the space containing X. If X is not compact, take the larger space to be the one point compactification $X^\bullet$. We know that X is dense in the latter, and if X is connected it follows that its closure in $X^\bullet$, which is $X^\bullet$ itself, must also be connected.■

(ii) FALSE. Let $f : [0,1] \to S^1$ be the map sending t to $\exp 2\pi i t$. Then f is continuous and onto, the domain of f is contractible, and the codomain of f has a nontrivial fundamental group.■

(iii) TRUE. Take a closed disk of sufficiently large radius which contains the given compact subset. The fundamental group of this disk is trivial, so the map of fundamental groups must be trivial.■

4. It suffices to show that the straight line homotopy from the identity map to the constant map with value p always takes values in $\cup_\alpha K_\alpha$. But if X lies in the latter and we choose β such that $x \in K_\beta$, then the line segment joining x to p will lie in K_β by convexity. Therefore if $x \in \cup_\alpha K_\alpha$, then the line segment joining x to p is entirely contained in the latter set. In other words, the homotopy inverse to the inclusion is given by the constant map with value p, and a homotopy from the identity to the constant map with value p is given by the straight line homotopy, whose image is entirely contained in $\cup_\alpha K_\alpha$. Therefore $\{p\}$ is a strong deformation retract of $\cup_\alpha K_\alpha$.■

5. By Theorem VIII.2.11 in the addendum document, it suffices to show that the fundamental groups of X and $X \times S^1$ are not isomorphic. We are given that the fundamental group of X is a finitely generated abelian group π, and it follows that the fundamental group of $X \times S^1$ must be isomorphic to $\pi \times \mathbb{Z}$. Since π and $\pi \times \mathbb{Z}$ are not isomorphic if π is finitely generated and abelian, the theorem implies that X and $X \times S^1$ cannot be homeomorphic.■