Math 205A
Fall 2014

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Additional files on free products

There is an alternate construction for the free product of an indexed family of groups (Theorem IX.2.4 in `fundgp-notes.pdf`) which corresponds to the approach in the lectures. The argument is based upon the Universal Mapping Property for free groups and the following Universal Mapping Property for quotient group projections:

PROPOSITION. *Let $f : G \to H$ be a group homomorphism, and let $L \subset G$ be a normal subgroup such that $f|L$ is trivial. Then there is a unique homomorphism $f^* : G/L \to H$ such that $f = f^* \circ q$, where $q : G \to G/L$ is the quotient projection.*

The proof of this is fairly straightforward. We provisionally define $f^*(gL) = f(g)$ and verify this is well defined because $g_1L = g_2L$ implies $g_1 = g_2y$ for some $y \in L$, so that

$$f(g_1) \quad = \quad f(g_2y) \quad = \quad f(g_2)f(y) \quad = \quad f(g_2) \cdot 1 \quad = \quad f(g_2)$$

because $f(y) = 1$ for $y \in L$. By construction we have $f^* \circ q(g) = f^*(gL) = f(g)$ for all $g \in G$, and the chain of equations

$$f^*(g_1L \cdot g_2L) \quad = \quad f^*\big((g_1g_2)L\big) \quad = \quad f(g_1g_2) \quad = \quad f(g_1) \cdot f(g_2) \quad = \quad f^*(g_1L) \cdot f^*(g_2L)$$

shows that f^* is a homomorphism. This proves the existence half of the result.

To prove uniqueness, suppose that $h : G/L \to H$ satisfies the analogous condition $h \circ q = f$. Then for each $gL \in G/L$ we have $h(gL) = h \circ q(g) = f(g)$, and since $f(g) = f^*(gL)$ it follows that $h(gL) = f^*(gL)$ for all cosets gL; in other words, we have $h = f^*$.■