Going beyond Mathematics 205A

Not surprisingly, there are many different ways in which the mathematical sciences build upon the material in a first graduate topology course, so we shall limit and organize our discussion with three basic themes:

Subsequent courses in the sequence. Topology is basically a geometrical subject, and the subsequent courses in the sequence (Mathematics 205B and 205C) reflect this fact, both in their basic problems and in the techniques which are employed.

Mathematics 205A covers Part I of Munkres and about half of Part II. Material from the rest of Part II is covered in the first weeks of Mathematics 205B. Exact coverage depends upon the specific instructor, but in the Winter 2014 course the main topics were finishing the material on covering spaces in Munkres and applying the methods of Part I I to a special class of spaces called ***finite graphs*** or ***finite graph complexes***. The latter arise naturally in many areas of mathematics and in its applications to other subjects.

One can summarize Part II of Munkres by saying that it defines and studies certain algebraic objects which are algebraic "pictures" of $1-$ dimensional configurations in a topological space; after covering this material, Mathematics 205B begins the definition and study of similar algebraic "pictures" of higher dimensional configurations. For example, if U is an open subset of $\mathbb{R}^3$, then the $2-$ dimensional configurations turn out to be closed surfaces and similar objects in U. One early and easily stated objective of such algebraic constructions is to verify a fact that one would probably expect: ***Two Euclidean (or Cartesian coordinate) spaces of different dimensions are not topologically equivalent.*** Once again, exact coverage depends upon the specific instructor, but in the Winter 2014 course the main topics were (1) defining a sequence of abelian groups called ***homology groups*** (for certain spaces built out of fairly simple pieces) indexed by the nonnegative integers, where the n^{th} group is formed from $n-$ dimensional configurations, (2) studying the problem of extending these definitions to arbitrary topological spaces and relating the $1-$ dimensional homology group to the fundamental group, (3) applying these tools to prove some basic facts which one would expect but are deceptively hard to prove mathematically. The topological inequivalence of Euclidean spaces with differing dimensions is one example, and another is the ***Jordan Curve Theorem***, which states that if C is a simple closed curve in $\mathbb{R}^2$ then its complement has two connected components, and C is the boundary of each component. More details on the content of the Winter 2014 course can be found in the directory http://math.ucr.edu/~res/math205B-2012.

One way to motivate 205C is to say that it describes some extra structure on topological spaces which suffices to define a viable concept of smoothly parameterized curves. A parallel motivation is that it provides an abstract setting for studying questions about curves and surfaces which arise in classical differential geometry. More precisely and abstractly, the objective is to develop a setting similar to point set topology in which one can work with objects which not only have a notion of continuity, but also have a compatible notion of differentiation; open subsets in Euclidean space are the basic models that we wish to generalize, and the main objects of interest are often called ***differentiable manifolds*** (for more details, see the ***Wikipedia*** link at the end of this document). In this course, there are also crucial ties to linear algebra and its generalizations, abstract multivariable calculus, and fundamental results on the theory of solutions to ordinary differential equations which go beyond the standard lower level undergraduate courses. Material from **205C** plays fundamentally important roles in many areas of topology, geometry and analysis (and to a lesser extent also algebra), and it is also fundamentally important in many areas of physics (among other things, it provides the mathematical framework for relativistic space – time).

Further topics in general topology. Topological spaces arise naturally in nearly every branch of the mathematical sciences, and in order to keep the discussion within reasonable bounds we shall concentrate on some areas outside of topology and geometry where the impact of general topology is most pronounced. A few online references are given below. The first two involve D. Rusin's wide ranging online overview of present day mathematics, and the third is the comprehensive site at York University (in Canada, located near Toronto) devoted to general topology.

http://www.math.niu.edu/~rusin/known-math/index/54-XX.html

http://www.math-atlas.org/

http://at.yorku.ca/topology /

Further methods and results in general topology play a crucial role in ***functional analysis***, which is an abstract study of large spaces of functions largely aimed at studying questions like solutions to (ordinary and partial) differential equations and good approximations to functions. Another branch of analysis in which topological spaces and their refinements play a key role is ***measure theory***. Topological spaces also figure importantly in some foundational areas. One example is the subject between general topology and mathematical logic which is called ***set – theoretic topology***, and another is the appearance of topological spaces in certain issues from ***theoretical computer science***. Further information on the latter is included near the end of Section **VI.3** in the course notes. For the sake of convenience, here are online references for the other topics mentioned in this paragraph:

http://en.wikipedia.org/wiki/Functional_analysis

http://en.wikipedia.org/wiki/Measure_theory

http://en.wikipedia.org/wiki/Set-theoretic_topology

Some general comments on the use of ***Wikipedia*** articles as authoritative references appear in the online file

http://math.ucr.edu/~res/math205A/aabInternetresources.pdf

and further references for (generally very reliable) information on mathematical topics through the level of current research are given below:

http://mathworld.wolfram.com

http://en.wikipedia.org/wiki/Main_Page

http://planetmath.org

Further topics involving fundamental groups. This will be more limited because the subject matter is more specialized and only half of the topic is covered in **205A.** There are many areas of topology, geometry and complex analysis where fundamental groups play an important role, and some will arise in the continuation of the subject in **205B.** Another way of noting the significance of the fundamental group is that there is a large, important class of objects called ***aspherical spaces*** for which the fundamental group completely determines the homotopy type of the space. The defining conditions for such a space are **(1)** points have neighborhood bases which are locally contractible, **(2)** the space itself has a contractible covering space. There are large classes of aspherical spaces for which one can even retrieve information about the topological structure of the space from its fundamental group. Here is one example: The methods and results of Unit **VIII** show that the n - dimensional torus T^n is aspherical, and in fact several breakthrough results from the past **65** years combine to imply that T^n is uniquely characterized up to homeomorphism by a few simple conditions: Namely, it is a Hausdorff space, its fundamental group is a free abelian group of rank n and it has a covering space which is homeomorphic to $\mathbb{R}^n$. Additional information on aspherical spaces can be found in the articles listed below (however, the second and third are meant as background for the remaining one). The fourth item touches on the role of aspherical spaces in the study of Riemannian geometry; the last two items involve more advanced material, but they do provide examples in which certain classes of topological spaces can be retrieved from their fundamental groups.

http://en.wikipedia.org/wiki/Aspherical_space

http://en.wikipedia.org/wiki/Differentiable_manifold

http://en.wikipedia.org/wiki/Riemannian_geometry

http://en.wikipedia.org/wiki/Cartan-Hadamard_theorem

http://www.mit.edu/~jamess/honoursfiles/honoursthesis.pdf

http://en.wikipedia.org/wiki/Mostow_rigidity