II. 1

Origins of point set topology

Describe properties of subsets of $\mathbb{R}^n$ which are relevant to calculus (open regions, open disks, ...) — More qualitative than geometry

see also
ps top-motiv.pdf

Distance between two points

Very useful for abstracting properties

$d(x,y) \geq 0$, $=$ iff $x = y$, $d(x,y) = d(y,x)$

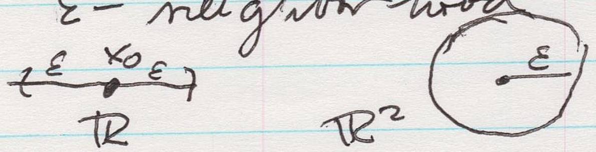
$d(x,z) \leq d(x,y) + d(y,z)$ $\underset{\sim\sim}{\Delta \leq}$

Examples $\mathbb{R}^n$, discrete $\underset{\sim\sim\sim}{d = 0 \text{ or } 1}$ on A

subsets of $\mathbb{R}^n$, cont. fcns. on $[0,1]$

$d(f,g) = \max_t |f(t) - g(t)|$

ε-neighborhood $\{d(y,x_0) < \varepsilon\}$



etc. $N_\varepsilon(x_0)$

Open sets (domains) If $x_0 \in U$ then some $N_\varepsilon(x_0) \subseteq U$.

Need for partial differentiation

Topological structures

Axiomatize basic properties of open sets

ϕ, X open in X, closed under $\begin{cases} \text{all unions} \\ \text{finite } \cap\text{'s.} \end{cases}$

Non metric example - Indiscrete space

ϕ, X = only open subsets.

<u>T_1 property</u> $x \in X \Rightarrow X - \{x\}$ open.

<u>T_2 (Hausdorff) property</u> $u \neq v \in X \Rightarrow$ have disjoint U, V open s.t. $u \in U$ & $v \in V$.

An easy result: Lemma II. 1.4, p. 15.

Topology generated by a family of subspaces

$\mathcal{Q} \subseteq P(X)$ set of subsets $\Rightarrow$

topology generated by $\mathcal{Q}$ = unions of finite intersections of elts in $\mathcal{Q}$ plus ϕ, X.

$\mathcal{B}$ = base $\Leftrightarrow$ topology is generated by $\mathcal{B}$ AND all open sets are unions of elts in $\mathcal{B}$.

<u>Example</u> X metric $\mathcal{B}$ = ε-disks (neighbor-
$\frac{1}{n}$ disks (hoods)

Subspace topology $A \subseteq X$

all $U \cap A$, U open in X.

X metric $\Rightarrow$ subspace topology = metric topology

$$\left[N_\varepsilon(x; A) = N_\varepsilon(x) \cap A \right].$$

II.2

Closed* subsets of $\mathbb{R}^n$ also important

(* under taking limits of sequences).

"interesting" ~~"Good"~~ seqs. Nontriviality condition for sequences: Throw out all a_n equal to limit value L.

See Prop. II.2.8 et seq

Unfortunately, topological spaces don't have enough structure to generalize $\lim_{n\to\infty} a_n = L$ decently. So we need to reformulate. Key result:

Prop. II.2.2 X metric, $A \subseteq X$, $y \in X$. TFAE

① $y = \lim_{n\to\infty} a_n$, where $a_n \neq y$ all n

② For every open nbhd U of y we have $(U - \{y\}) \cap A \neq \phi$.

Is meaningful in top spaces!!

Two things to prove ① $\Rightarrow$ ②

② $\Rightarrow$ ①.

If ② holds, say y is a limit point of A.
$L(A)$ = set of all limit points.

Thm II.2.3 TFAE

① $X - A$ is open
② $L(A) \subseteq A$.
} Say A is closed if these hold.

Proof in notes (important to know)

Family of closed subsets "closed" under { intersections, finite unions

Topology determined by closed subsets. (One of many alternate definitions).

$A \cup L(A)$ is closed in X. closure $\overline{A}$

Proof. (need to understand actively)

$\overline{A} \subseteq$ every closed set F such that $F \supseteq A$.

Interior. $\operatorname{Int} A$ = largest open set contained in A. Formally, $\operatorname{Int} A = X - \overline{X - A}$

(numerous exercises deal with equivalent descriptions). Boundary See exercises.