$\underline{\underline{\text{II}.3}}$

continuous fcn. $\forall x\ \forall \varepsilon > 0\ \exists \delta_x > 0$ s.t.

$$d(x,y) < \delta_x \Rightarrow d(f(x), f(y)) < \varepsilon.$$

<u>Thm. ~~III~~ 3.1</u> $f: X \to Y$ continuous $\Longleftrightarrow$ inverse images of open sets are open.

Proof.

This property extends to top. spaces.

Same true for "closed" replacing "open."

<u>Thm. II. 3.2</u> other characterizations of continuity.

<u>Formal properties of continuity</u>

III.3.3 $X \xrightarrow{f} Y \xrightarrow{g} Z$ $f + g$ cont. $\Rightarrow$ so is composite $g \circ f$

III.3.4 $A \subseteq X$ w/ subspace topology $\Rightarrow i: A \xrightarrow{\subseteq} X$ cont.

III.3.5 <u>Corestriction property</u> $f: X \to Y$ cont.

$j: B \subseteq Y$, $f[X] \subseteq B \Rightarrow f = j \circ g$, $g: X \to B$ unique cont map

$X = \cup A_\alpha \quad f: X \to Y, \ f_\alpha = f|A_\alpha = f \circ i_\alpha.$

III.3.10 ① A_α open. Then f cont. $\Leftrightarrow$ all f_α cont.

② A_α closed, FINITE FAMILY. Same conclusion

<u>HOMEOMORPHISM</u> $f: X \to Y$ 1-1 onto

Both f and f^{-1} continuous.

<u>Examples</u> "identity": $(X, \text{discrete}) \to (X, \text{indiscr})$

1-1 onto continuous, inverse isn't if $|X| \geq 2$.

$[0,1) \longrightarrow S^1 \quad t \to e^{2\pi i t}$.

$S^1 \to [0,1)$ not cont. inverse image of $[0,\frac{1}{2})$ isn't open in S^1. (why?)

$1_X : (X, \mathcal{T}) \to (X, \mathcal{T})$ homeo

f homeo $\Rightarrow$ so is f^{-1}

f, g composable homeos $\Rightarrow g \circ f$ homeo.

Alternate characterizations $f : X \to Y$ 1-1 onto cont.

f sends open sets to open sets
f sends closed sets to closed sets.

Cor: Homeo $\iff$ induces 1-1 correspondence of $\left\{\begin{matrix}\text{open}\\ \text{closed}\end{matrix}\right\}$ subsets.

Examples $f : \mathbb{R} \to \mathbb{R}$
$f(x) = x^{2n+1}$
$n \geq 0$ integer

Related concepts $f : X \to Y$
f is $\left\{\begin{matrix}\text{open}\\ \text{closed}\end{matrix}\right\}$ if $A \subseteq X \left\{\begin{matrix}\text{open}\\ \text{closed}\end{matrix}\right\} \Rightarrow$ so is $f[A] \subseteq B$.

Examples. 1_X is both
$x^2 : \mathbb{R} \to \mathbb{R}$ cont, $f[\mathbb{R}] = [0, \infty)$, not open
$\arctan x : \mathbb{R} \to \mathbb{R}$ cont. $f[\mathbb{R}] = (-\frac{\pi}{2}, \frac{\pi}{2})$ not closed
$\sin x$ is neither. $f[\mathbb{R}] = [-1, 1]$ not open

$\cup \left[2k\pi, (2k + \frac{1}{2})\pi\right]$ is closed (complement = union of intervals)

but image is $[0, 1)$, not closed.

Geometric
Metamathematical } properties of homeos.

(FINALLY) See homeomorphisms.pdf