II.4

Products of spaces

Formal construction but useful (MV calculus)

$X_1, \dots, X_n$ spaces (finite!)

Product topology = topology gen by $U_1 \times \dots \times U_n$, U_i open in X_i — these form a base (II.4.1).

II.4.2 $\left.\begin{matrix}\mathcal{B}_X \\ \mathcal{B}_Y\end{matrix}\right\}$ base for $\left\{\begin{matrix}X \\ Y\end{matrix}\right\} \Rightarrow$ all $\underset{\in \mathcal{B}_X}{U} \times \underset{\in \mathcal{B}_Y}{V}$ form a base for $X \times Y$. $\left[\begin{matrix}\text{SIMPLEST} \\ \text{CASE}\end{matrix}\right]$

II.4.4 Product topology = smallest topology s.t. $\pi_i : \prod X_j \to X_i$ (coord proj) continuous.

Proof

II.4.5 $f : Y \to \prod X_j$ map of sets $\begin{matrix}Y \\ X_j\end{matrix}$ t. spaces

f cont. $\Leftrightarrow$ all $\pi_j \circ f$ cont.

II.4.6 F_i closed in $X_i \Rightarrow \prod F_j$ closed in $\prod X_j$.

II.4.7 $\overline{\prod A_i} = \prod \overline{A_i}$.

II.4.8 π_j is open.

$f_i : X_i \to Y_i$ cont. $\Rightarrow$ so is $\prod f_i$.

$\prod f_i (x_1, \ldots, x_n) = (f_1(x_1), \text{etc.})$

Products and metrics

X_1, X_2 metric. Three metrics on $X_1 \times X_2$

$d_\infty(x, y) = \max_i d_i(x_i, y_i)$ MAX

$d_1 \quad = \text{sum}.$ TAXICAB

$d_2 \quad = \sqrt{d_1(x_1, y_1)^2 + d_2(x_2, y_2)^2}$

PYTHAGOREAN

$d_0 \le d_2 \le d_1 \le 2 \cdot d_2 \Rightarrow$

all maps $(X \times Y, d_p) \to (\quad , d_q)$

cont.

ε-disk in d_∞ metric $= N_\varepsilon(x_1) \times N_\varepsilon(x_2)$.

So all three metrics define product topology.

Product of $\frac{T_1}{T_2}$ Spaces is $\frac{T_1}{T_2}$

<u>Proof.</u>

Infinite products

base = open sets $\prod U_\alpha$ where all but finitely many U_α's $= X_\alpha$'s.

Smallest topology on $\prod X_\beta$ s.t. coord projs $\pi_\alpha : \prod X_\beta \to X_\alpha$ all cont.

Many but not all previous results go through.

(Putting a metric on $\prod X_\alpha$ is less simple, usually impossible.)

<u>Reminder from MV calculus</u>

$f : X \times Y \to Z$ cont. $\Rightarrow$ for each $a \in X$, $b \in Y$

$f\Big|_{\{a\}\times Y}^{X\times\{b\}}$ is cont. However the converse is false.