Details of the construction(s) in `def-retr-munkres362.pdf`

For the sake of completeness we shall provide details for the argument in the cited document. The first step is to give explicit definitions for the subsets depicted there.

> The set A is the union of the two circles, whose centers are the points $(\pm 1, 0)$ and whose radii are equal to 1; *i.e.*, the two circles are defined by the equations $(x \mp 1)^2 + y^2 = 1$.
>
> The set B is the union of the inner regions of the two circles in A with the centers of the circles deleted; *i.e.*, the two pieces of B are defined by the equations $0 < (x \mp 1)^2 + y^2 \leq 1$.
>
> The set C is vertical strip defined by $|x| \leq 2$ with the centers of the two circles deleted.
>
> The set D is the complement of the center points $(\pm 1, 0)$.

By Exercise 58.1 in Munkres, if X is a strong deformation retract of Y and Y is a deformation retract of Z, then X is a strong deformation retract of Z. Therefore, the proof that A is a strong deformation retract of D reduces to proving

(1) C is a strong deformation retract of D,

(2) B is a strong deformation retract of C, and

(3) A is a strong deformation retract of B.

As indicated in `def-retr-munkres362.pdf`, the homotopy inverse to $C \subset D$ is the map $\rho_1 : D \to C$ which is the identity if $|x| \leq 2$ and sends (x, y) to $(\mathrm{sgn}\,(x) \cdot 2, y)$ if $|x| \geq 2$, where $\mathrm{sgn}\,(x)$ is ± 1 depending upon whether x is positive or negative. Then $\rho_1|C$ is the identity, and if $i_1 : C \to D$ is the inclusion, then $i_1 \circ \rho_1$ is homotopic to the identity via a straight line homotopy along horizontal lines, and this homotopy is fixed on C. This yields (1).

To define the map(s) for the second step, let $h(t)$ be the continuous function on $[-2, 2]$ defined by $h(t) = \sqrt{1 - (t+1)^2}$ if $t \leq 0$ and $h(t) = \sqrt{1 - (t-1)^2}$ if $t \geq 0$ (so the graph is the union of the two upper semicircles in A). These definitions agree on the unique overlapping point where $t = 0$, so these formulas define a continuous function, and it follows that B is the subset of all points $(x, y) \in C$ such that $|y| \leq h(x)$. As indicated in `def-retr-munkres362.pdf`, the homotopy inverse to $B \subset C$ is the map $\rho_2 : C \to B$ which is the identity if $|y| \leq h(x)$, is equal to $(x, h(x))$ if $y \geq h(x)$, and is equal to $(x, -h(x))$ if $y \leq -h(x)$. Then $\rho_2|B$ is the identity, and if $i_2 : B \to C$ is the inclusion, then $i_2 \circ \rho_2$ is homotopic to the identity via a straight line homotopy along vertical lines, and this homotopy is fixed on B. This yields (2).

Finally, we need to define the map(s) for the third step. It will be convenient to use vector notation, so write a point in $\mathbb{R}^2$ as $\mathbf{v} = (x, y)$ and let $\mathbf{e}_1$ denote the unit vector $(1, 0)$. As indicated in `def-retr-munkres362.pdf`, the homotopy inverse to $A \subset B$ is the map $\rho_3 : B \to A$ given by

$$\rho_3(\mathbf{v}) \quad = \quad \mathbf{e}_1 \; + \; \frac{1}{|\mathbf{v} - \mathbf{e}_1|} \cdot (\mathbf{v} - \mathbf{e}_1) \quad \text{if} \quad x \quad \geq \quad 0\,,$$

$$\rho_3(\mathbf{v}) \quad = \quad -\mathbf{e}_1 \; + \; \frac{1}{|\mathbf{v} + \mathbf{e}_1|} \cdot (\mathbf{v} + \mathbf{e}_1) \quad \text{if} \quad x \quad \leq \quad 0\,.$$

The only point for which both conditions hold is $\mathbf{v} = \mathbf{0}$, and in this case both definitions of ρ_3 yield the vector $\mathbf{0}$. It follows that $\rho_3|A$ is the identity, and and if $i_3 : A \to B$ is the inclusion, then $i_3 \circ \rho_3$ is homotopic to the identity via a straight line homotopy (in order to justify this, one must check that the line segment joining $\mathbf{v}$ to $i_3 \circ \rho_3(\mathbf{v})$ does not pass through the points $\pm \mathbf{e}_1$). By construction this homotopy is fixed on A, and this yields (3).

As noted above, if we combine these steps we see that A is a strong deformation retract of D. ∎