More metrics on cartesian products

If $(X_i, \mathbf{d}_i)$ are metric spaces for $1 \leq i \leq n$, then in Section II.4 of the lecture notes we defined three metrics on $\prod_i X_i$ whose underlying topologies are the product topology. The purpose of this note is to explain how one can interpolate a continuous family of metrics between these examples; for each such metric, the underlying topology will be the product topology.

Throughout this discussion $p \geq 1$ will denote a fixed real number.

Let $x, y \in \prod_i X_i$, express them in terms of coordinates as $(x_1, \cdots, x_n)$ and $(y_1, \cdots, y_n)$ respectively, and define $\mathbf{d}_p$ from $(\prod_i X_i) \times (\prod_i X_i)$ to $\mathbb{R}$ as follows:

$$\mathbf{d}^{\langle p \rangle}(x, y) \quad = \quad \left(\sum_i \mathbf{d}_i(x_i, y_i)^p \right)^{1/p}$$

The cases where $p = 1$ or 2 were considered in the lecture notes.

It follows immediately that $\mathbf{d}^{\langle p \rangle}$ satisfies all the properties for a metric except perhaps the fundamentally important Triangle Inequality. The latter is in fact a consequence of the following basic result:

Minkowski's Inequality. *Suppose that we have $u, v \in \mathbb{R}^n$ and we write these vectors in coordinates as $(u_1, \cdots, u_n)$ and $(v_1, \cdots, v_n)$ respectively. Then we have*

$$\left(\sum_i |u_i + v_i|^p \right)^{1/p} \quad \leq \quad \left(\sum_i |u_i|^p \right)^{1/p} + \left(\sum_i |v_i|^p \right)^{1/p} \quad . \blacksquare$$

Here are some references for a proof of Minkowski's Inequality:

W. Rudin, Real and Complex Analysis. (Third Edition. Mc-Graw-Hill Series in Higher Mathematics.) *McGraw-Hill, Boston*-etc., 1987. ISBN: 0-07-054234-1.

`http://www.planetmath.org/encyclopedia/MikowskiInequality.html`

The incorrect spelling "Mikowski" needed to reach the `planetmath` link should be noted; the latter also gives further links to the proof of the inequality, the statement and proof of the closely related Hölder Inequality, and a statement and proof of the Young Inequality which can be used to prove Hólder's Inequality; in fact, one generally begins by proving Hölder's Inequality (either as in the `planetmath` links or by some other means) and then derives Minkowski's inequality from Hölder's Inequality.

Hölder's Inequality. *Suppose that we have $u, v \in \mathbb{R}^n$ as above with $p > 1$, and that we choose $q > 1$ such that*

$$\frac{1}{q} + \frac{1}{p} \quad = \quad 1 \ .$$

Then we have

$$\left(\sum_i |u_i \cdot v_i| \right) \quad \leq \quad \left(\sum_i |u_i|^p \right)^{1/p} \cdot \left(\sum_i |v_i|^q \right)^{1/q} \quad . \blacksquare$$

The `planetmath` references also contain a sequence of links to Hölder's inequality and related facts which can be used bo give a self-contained proof of the two given inequalities and some other basic results.

Since each of the metrics $\mathbf{d}^{\langle p \rangle}$ for $p = 1, 2, \infty$ defines the product topology, it is natural to speculate that the same holds for all choices of p, and in fact this is true.

PROPOSITION. *For each $p \geq 1$, the topology determined by the metric $\mathbf{d}^{\langle p \rangle}$ is the product topology. Furthermore, the identity map from $(\prod_i X_i, \mathbf{d}^{\langle \alpha \rangle})$ to $(\prod_i X_i, \mathbf{d}^{\langle \beta \rangle})$ is uniformly continuous for all choices of α, β such that $1 \leq \alpha, \beta \leq \infty$.*

Proof. It suffices to prove the assertion in the second sentence, and the latter reduces to the special case where one of α, β is ∞; if we know the result in such cases, we can retrieve the general case using the uniform continuity of the identity mappings

$$\left(\prod_i X_i, \mathbf{d}^{\langle p \rangle} \right) \longrightarrow \left(\prod_i X_i, \mathbf{d}^{\langle \infty \rangle} \right) \longrightarrow \left(\prod_i X_i, \mathbf{d}^{\langle r \rangle} \right)$$

and the fact that a composite of uniformly continuous maps is uniformly continuous.

The uniform continuity statements are direct consequences of the following inequalities for nonnegative real numbers u_i for $1 \leq i \leq n$:

$$\max_i \{ u_i \} \quad \leq \quad \left(\sum_i u_i^p \right)^{1/p} \quad \leq \quad n \cdot \max_i \{ u_i \}$$

One can then apply the argument in the notes to show that the identity maps

$$\left(\prod_i X_i, \mathbf{d}^{\langle \infty \rangle} \right) \longrightarrow \left(\prod_i X_i, \mathbf{d}^{\langle p \rangle} \right) \longrightarrow \left(\prod_i X_i, \mathbf{d}^{\langle \infty \rangle} \right)$$

are uniformly continuous (and in fact the δ corresponding to a given ε can be read off explicitly from the inequalities!), and of course all composites of maps from this diagram are also uniformly continuous.■

The limiting case

The following result is the motivation for setting $\mathbf{d}_\infty$ equal to the maximum distance between coordinates:

PROPOSITION. *In the setting above we have*

$$\mathbf{d}^{\langle \infty \rangle} \quad = \quad \lim_{p \to \infty} \mathbf{d}^{\langle p \rangle} \ .$$

Proof. This reduces immediately to proving the following result: *If $u \in \mathbb{R}^n$ as above then*

$$\max_i \{ |u_i| \} \quad = \quad \lim_{p \to \infty} \left(\sum_i |u_i|^p \right)^{1/p} \ .$$

Let M denote the expression on the left hand side, and for each $p > 1$ let Y_p denote the value of the expression whose limit we wish to find. Clearly $M \leq Y_p$ for all p because M is obtained by deleting all but one summand from Y_p. However, since $|u_i| \leq M$ for all i, we also have $Y_p \leq (n \cdot M^p)^{1/p} = M \cdot n^{1/p}$. Now the limit of the right hand side as $p \to \infty$ is equal to M, and thus we have sandwiched Y_p between two expressions, one of which is equal to M and the other of which has a limit equal to M. It follows that the limit of Y_p is also equal to M, which is exactly the claim in the proposition.■

If one graphs the set of all points in $\mathbb{R}^2$ whose p-distance from the origin is equal to 1 for various values of $p \geq 1$, the result is a collection of closed curves centered at the origin such that the area enclosed by the curve increases with p and the limit of these curves is the boundary of the square whose vertices are the elements of the set $\{\pm 1\} \times \{\pm 1\}$.

Refined estimates

It is straightforward to check that the solid unit disk in $\mathbb{R}^2$ with respect to the $\mathbf{d}^{\langle 1 \rangle}$ metric is the solid square region whose vertices are $(\pm 1, 0)$ and $(0, \pm 1)$, while the solid unit disks in $\mathbb{R}^2$ with respect to the $\mathbf{d}^{\langle 2 \rangle}$ and $\mathbf{d}^{\langle \infty \rangle}$ metrics are (respectively) the usual round unit disk and the solid square $[-1,1] \times [-1,1]$. In particular, if $\alpha < \beta$ then the unit disk with respect to $\mathbf{d}^{\langle \beta \rangle}$ strictly contains the unit disk with respect to $\mathbf{d}^{\langle \alpha \rangle}$. Likewise, the corresponding unit disks in $\mathbb{R}^3$ are the solid octahedral region with vertices $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$ and $(0, 0, \pm 1)$, and the solid cube $[-1, 1]^3$, with each set properly contained in the next. There are analogous statements in all higher (finite) dimensions. We shall generalize these observations to arbitrary metrics $\mathbf{d}^{\langle \alpha \rangle}$ and $\mathbf{d}^{\langle \beta \rangle}$ where $1 \leq \alpha < \beta \leq \infty$.

THEOREM. *Let α and β satisfy $1 \leq \alpha < \beta \leq \infty$, and suppose that $n \geq 2$. Then the solid unit disk in $\mathbb{R}^n$ with respect to the metric $\mathbf{d}^{\langle \beta \rangle}$ strictly contains the analogous disk with respect to the metric $\mathbf{d}^{\langle \alpha \rangle}$.*

Of course, if $n = 1$ then all the analogous disks are the same.

Proof. As usual, let $|\cdots|_\alpha$ and $|\cdots|_\beta$ denote the α- and β-norms on $\mathbb{R}^n$. The first step is to show that if $\beta > \alpha$ and $|x|_\alpha \leq 1$ then we also have $|x|_\beta \leq 1$.

Since $|x|_p$ only depends upon the absolute values of the coordinates of x, it suffices to consider the case where all the coordinates of x are nonnegative. Furthermore, since $|x|_\alpha = |x|_\beta$ if x is a multiple of a unit vector, it will suffice to prove the result when at least two of the coordinates of x are nonzero, in which case it follows that all the (absolute values of the) coordinates are all strictly less than 1.

CLAIM: *If x is as above and $|x|_\alpha \leq r \leq 1$ then for all $\beta > \alpha$ we have $|x|_\beta < r$.*

To prove the claim, consider the function

$$N_x(p) \quad = \quad \left(x_1^p \; + \; \cdots \; + \; x_k^p\right)^{(1/p)}$$

where $p \geq 1$. For all $p \geq 1$ we have

$$N'_x(p) \quad = \quad \frac{1}{p} \left(\sum_{i=1}^{n} \; x_i^p \right)^{\frac{1-p}{p}} \quad \cdot \sum_{x_j \neq 0} \; (\log_e \, x_j) \, x_j^p$$

and the right hand side is negative because (i) the values x_i^p are all nonnegative but less than 1, (ii) at least two of the numbers x_i are positive, so that the associated logarithmic coefficients are negative and the terms x_i^p are positive. Therefore by the Mean Value Theorem we know that N_x is a strictly decreasing function for $p \geq 1$. This immediately proves the claim if $\beta < \infty$. In the remaining case where $\beta = \infty$ we know that $|x|_\infty = \lim_{p \to \infty} \; N_x(p)$ and since N_x is strictly decreasing it follow sthat the limit value is strictly less than $|x|_p$ for all p such that $1 \leq p \leq \infty$. This completes the proof of the claim.

In particular, the preceding discussion shows that if $|x|_\alpha \leq 1$ then $|x|_\beta \leq 1$ so that the unit disk with respect to $\mathbf{d}^{\langle\alpha\rangle}$ is contained in the unit disk with respect to $\mathbf{d}^{\langle\beta\rangle}$. To prove the statement about strict containment, let x be the vector whose first two coordinates are $2^{-\alpha}$ and whose remaining coordinates are zero, so that $|x|_\alpha = 1$. If we let $b = |x|_\beta$, then $b > 0$ and by the preceding discussion we know that $b < 1$. The basic properties of norms now imply that $|b^{-1}\,x|_\beta = 1$ while $|b^{-1}\,x|_\alpha = b^{-1} > 1$, and therefore it follows that $b\,x$ lies in the unit disk with respect to $\mathbf{d}^{\langle\beta\rangle}$ but not in the unit disk with respect to $\mathbf{d}^{\langle\alpha\rangle}$, proving that the unit disk with respect to the first metric strictly contains the unit disk with respect to the second.∎

A figure illustrating the first quadrant portions of some unit disks with respect to $\mathbf{d}^{\langle p\rangle}$ metrics appears in the file `dpunitdisks.pdf`.

The preceding results for the $\mathbf{d}^{\langle p\rangle}$ metrics on $\mathbb{R}^n$ generalize immediately to other products. It will be convenient to introduce a the following property for metric spaces.

Definition. Let $\varepsilon > 0$. A metric space $(X, \mathbf{d})$ is said to be ε-**weakly saturated** at $x \in X$ if for all $\delta \in [0, \varepsilon]$ there is a point $y \in X$ such that $\mathbf{d}(x, y) = \delta$.

Clearly a normed vector space determines a weakly saturated metric with respect to every point, but a set with the discrete metric does not. In practice, many interesting examples of spaces satisfy weak saturation conditions. For example, if the underlying topological space X is connected in the sense of Unit III and contains more than one point, then for each $x \in X$ one can find some $\varepsilon(x) > 0$ such that x is $\varepsilon(x)$-weakly saturated at x; a proof is given at the end of this document.

THEOREM. *Suppose that we are given metric spaces $(X_i, \mathbf{d}_i)$ for $1 \leq i \leq n$, let $1 \leq \alpha < \beta \leq \infty$, and let $\mathbf{d}_\alpha$ and $\mathbf{d}_\beta$ be the associated product metrics on $\prod_i X_i$. Then for all x and y in the product we have the relation*

$$\mathbf{d}^{\langle\alpha\rangle}(x,y) \leq r \leq 1 \qquad \Longrightarrow \qquad \mathbf{d}^{\langle\beta\rangle}(x,y) \leq r\ .$$

Furthermore, if $x = (x_1, \cdots, x_n)$ and for each i the metric for X_i is 1-weakly saturated at x_i, then there are points $x, y \in \prod_i X_i$ such that $\mathbf{d}^{\langle\beta\rangle}(x,y) = 1$ but $\mathbf{d}^{\langle\alpha\rangle}(x,y) > 1$.

Proof. The displayed relation follows immediately from the preceding theorem. To prove the second, it suffices to find a point $y = (y_1, \cdots, y_n)$ such that $\mathbf{d}_1(x_1, y_1) = \mathbf{d}_2(x_2, y_2) = 2^{-\beta}$ and $y_i = x_i$ for all $i \geq 3$ (this is an empty condition if $n = 2$). Then the argument in the previous theorem implies that the β-distance from x to y is 1 but the α-distance is strictly greater than 1.∎

Saturation and connectedness

We shall conclude by giving simple conditions under which a metric satisfies saturation hypotheses at each point. This discussion involves the concept of connectedness, which is introduced in Unit III.

The main facts about connected spaces that we shall need are (i) a subset C of the real line is connected if and only if for all $x, y \in C$ such that $x < y$ and all z such that $x < z < y$ we have $z \in C$, (ii) if $f : X \to Y$ is continuous and X is connected then $f[X]$ is also connected. Given a metric space $(X, \mathbf{d})$, we shall also need and use the basic continuity properties of functions defined in terms of $\mathbf{d}$.

PROPOSITION. *Let $(X, \mathbf{d})$ be a connected metric space consisting of more than one point.*

(i) For each $x \in X$ there is some $\varepsilon_x > 0$ such that X is ε_x-weakly saturated at x.

(ii) *If the metric* $\mathbf{d}$ *is unbounded (in other words, its image is not a bounded subset of* $\mathbb{R}$*), then for all* $x \in X$ *and all* $\varepsilon > 0$ *the metric space* $(X, \mathbf{d})$ *is* ε*-weakly saturated at* x*.*

Proof. (i) Let $x \in X$ be arbitrary, and let $y \in X$ be such that $y \neq x$. Take $\varepsilon_x = \mathbf{d}(x, y) > 0$, and let $h : X \to \mathbb{R}$ be the (continuous) function $h(u) = \mathbf{d}(x, u)$. Then we know that $h[X]$ is a connected subset of $\mathbb{R}$. Since $h(x) = 0$ and $h(y) = \varepsilon_x$, by connectedness we know that $h[X]$ must contain the entire interval $0, \varepsilon_x]$.

(ii) By the preceding argument it suffices to show that if $\varepsilon > 0$ and $x \in X$, then there is some point $y \in X$ such that $\mathbf{d}(x, y) \geq \varepsilon$. — Suppose this is false for some particular $x \in X$ and $\varepsilon > 0$, so that $\mathbf{d}(x, y) < \varepsilon$ for all y. The unboundedness assertion for the metric implies that there are points $u, v \in X$ such that $\mathbf{d}(u, v) > 2\varepsilon$. By previously derived consequences of the Triangle Inequality we then have

$$\mathbf{d}(x, u) \quad \geq \quad \mathbf{d}(u, v) \; - \; \mathbf{d}(x, v) \quad \geq \quad 2\varepsilon \; - \; \varepsilon \quad = \quad \varepsilon$$

which contradicts our hypothesis that $\mathbf{d}(x, w) < \varepsilon$ for all $w \in X$. This yields the statement at the beginning of the paragraph.■